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STRUCTURES OF SU-RANK OMEGA WITH A DENSE INDEPENDENT SUBSET OF GENERICS

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Abstract. Extending the work done in [5, 9] in the o-minimal and geometric settings, we study expansions of models of a supersimple theory of SU-rank ω with a "dense codense" independent collection H of element of rank ω, where density of H means it intersects any definable set of SU-rank omega. We show that under some technical conditions, the class of such structures is first order. We prove that the expansion is supersimple and characterize forking and canonical bases of types in the expansion. We also analyze the effect these expansions have on one-basedness and CM-triviality. In the one-based case, we describe a natural "geometry of generics modulo H" associated with such expansions and show it is modular.

1. Introduction

There are several papers that deal with expansions of simple theories with a new unary predicate. For example, there is the expansion with a random subset [8] that gives a case where the new theory is again simple and forking remains the same, in contrast to the case of lovely pairs [2, 15], where the pair is usually much richer and the complexity of forking is related to the geometric properties of the underlying theory [15].

In [5] the first and the third authors studied, in the setting of geometric structures, adding a predicate for an algebraically independent set H which is dense and codense in a model M (meaning every non-algebraic formula in a single variable has a realization in H and a realization generic over H and its parameters). The paper generalized ideas developed in the framework of o-minimal theories in [9]. The key tool used in [5] was that the closure operator acl has the exchange property and thus gives a matroid that interacts well with the definable subsets. A special case under consideration was SU-rank one theories, where forking independence agrees with algebraic independence. In this stronger setting the authors characterized forking and gave a description of canonical bases in the expansion. As in the lovely pair case, the complexity of forking is related to the underlying geometry of the base theory T.

In this paper we start with a theory T that has SU-rank omega and we use the closure operator associated to the weight of generic types, namely for $M \models T$, $a \in M$, $A \subset M$, we have $a \in \mathrm{cl}(A)$ if $SU(a/A) < \omega$. This closure operator has the exchange property and many of the results obtained in [5] can be proved in the new framework: we expand M by a new predicate consisting in a cl-dense cl-codense

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family of independent generics (see Definition 2.3). In particular, the extension is supersimple and we get a clear description of canonical bases in the expansion, up to interalgebraicity (see Proposition 5.6).

In the special case where the theory of M is superstable with a unique type of U-rank ω, the predicate H is a Morley sequence of generics; this case is related to the work done in [1]. Our work is also related to work of Fornasiero on lovely pairs of closure operators [10].

Of special interest is the effect of our expansion on the geometric complexity, namely the ampleness hierarchy. Following the ideas of [7], we show that the expansion preserves CM-triviality, but one-basedness is preserved only in the trivial case.

We then use this expansion to study the underlying geometry of the closure operator localized in H. We show that if T is a one-based supersimple theory of SU-rank ω, (N, H) a sufficiently (e.g. $|T|^+$-) saturated H-structure, then the localized closure operator $\mathrm{cl}(- \cup H)$ is modular and its associated geometry is a disjoint union of projective geometries over division rings and trivial geometries.

This paper is organized as follows. In section 2 we define H-structures associated to models M of a theory T. We show that two H-structures associated to the same theory are elementary equivalent and call T^{ind} this common theory. Finally we prove that that under some technical conditions (elimination of the quantifier $\exists^{large}$ and the type definablity of the predicates $Q_{\varphi,\psi}$) the saturated models of T^{ind} are again H-structures.

In section 3 we study four different examples of theories of SU-rank ω: differentially closed fields, vector spaces with a generic automorphism, H-pairs and lovely pairs of geometric theories. In each case we show the corresponding theory of H-structures is first order.

In section 4 we analyze the definable sets in the expansion, we prove that every definable set is a boolean combination of old formulas bounded by existential quantifiers over the new predicate. In section 5 we characterize forking in the expansion and characterize canonical bases. In section 6 we study the question of preservation of one-basedness and CM-triviality under our expansion. Finally in section 7 we study the geometry of $\mathrm{cl}(- \cup H)$.

2. H-structures: definition and first properties

Let T be a simple theory of SU-rank ω. Let H be a new unary predicate and let $\mathrm{cl}_H = \mathrm{cl} \cup \{H\}$. Let T' be the $\mathcal{L}_H$-theory of all structures (M, H), where $M \models T$ and $H(M)$ is an independent subset of generic elements of M, that is, all elements have SU-rank ω. Note that saying that $H(M)$ is an independent collection of generics is a first order property, it is simply the conjunctions of formulas of the form $\neg\varphi(x_1, \ldots, x_n)$, where $SU(\varphi(x_1, \ldots, x_n)) < \omega n$.

For $M \models T$, $A \subset M$ and $b \in M$, we write $b \in \mathrm{cl}(A)$ and say that b is *small* over A if $SU(b/A) < \omega$. By the additivity properties of SU rank we have that $\mathcal{L}$ gives a pregeometry on M. We write $dim_{\mathrm{cl}}(\varphi(x_1, \ldots, x_n)) = n$ and say that $\varphi(x_1, \ldots, x_n)$ is *large* if $SU(\varphi(x_1, \ldots, x_n)) = \omega n$

We will assume that for every formula $\varphi(x, \vec{y})$ there is a formula $\psi(\vec{y})$ such that for any $\vec{a} \in M$ $\varphi(x, \vec{a})$ is large if and only if $\psi(\vec{a})$. We write $\exists^{large}\varphi(x, \vec{y})$ if $\psi(\vec{y})$ holds.

There is a strong analogy to what happens in geometric theories (see [3]), we change the pregeometry acl for the pregeometry $\mathcal{L}$ and the quantifier $\exists^\infty$ for the quantifier $\exists^{large}$.

Notation 2.1. *Let $(M, H(M)) \models T'$ and let $A \subset M$. We write $H(A)$ for $H(M) \cap A$.*

Notation 2.2. *Throughout this paper* independence *means independence in the sense of T and we use the familiar symbol $\mathop{\smile\!\!\!\!\!\mid}$. We write $\mathrm{tp}(\vec{a})$ for the $\mathcal{L}$-type of a and* dcl, acl *for the definable closure and the algebraic closure in the language $\mathcal{L}$. Similarly we write $\mathrm{dcl}_H, \mathrm{acl}_H, \mathrm{tp}_H$ for the definable closure, the algebraic closure and the type in the language $\mathcal{L}_H$.*

Definition 2.3. We say that $(M, H(M))$ is an *H-structure* if

(1) $(M, H(M)) \models T'$
(2) (Density/coheir property) If $A \subset M$ is finite and $q \in S_1(A)$ is the type of a generic element (of SU-rank ω), there is $a \in H(M)$ such that $a \models q$.
(3) (Co-density/extension property) If $A \subset M$ is finite and $q \in S_1(A)$, there is $a \in M$, $a \models q$ and $a \mathop{\smile\!\!\!\!\!\mid}_A H(M)$.

Lemma 2.4. *Let $(M, H(M)) \models T'$. Then $(M, H(M))$ is an H-structure if and only if:*

(2') *(Generalized density/coheir property) If $A \subset M$ is finite and $q \in S_n(A)$ has SU-rank ωn, then there is $\vec{a} \in H(M)^n$ such that $\vec{a} \models q$.*
(3') *(Generalized co-density/extension property) If $A \subset M$ is finite dimensional and $q \in S_n(A)$, then there is $\vec{a} \in M^n$ realizing q such that $\mathrm{tp}(\vec{a}/A \cup H(M))$ does not fork over A.*

Proof. We prove (2') and leave (3') to the reader. Let $\vec{b} \models q$, we may write $\vec{b} = (b_1, \ldots, b_n)$. Since $(M, H(M))$ is an H-structure, applying the density property we can find $a_1 \in H(M)$ such that $\mathrm{tp}(a_1/A) = \mathrm{tp}(b_1/A)$. Let $q(x, b_1, A) = tp(b_2, b_1, A)$ and let $A_1 = A \cup \{a_1\}$. Finally consider the type $q(x, a_1, A)$ over A_1, which is the type of a generic element. Applying the density property we can find $a_2 \in H(M)$ such that $\mathrm{tp}(a_2, a_1/A) = \mathrm{tp}(b_2, b_1/A)$. We continue inductively to find the desired tuple $(a_1, a_2, \ldots, a_n)$. □

Note that if $(M, H(M))$ is an H-structure, the extension property implies that M is $\aleph_0$-saturated.

Definition 2.5. Let A be a subset of an H-structure $(M, H(M))$. We say that A is *H-independent* if A is independent from $H(M)$ over $H(A)$.

Lemma 2.6. *Any model M of T with a distinguished independent subset $H(M)$ can be embedded in an H-structure in an H-independent way.*

Proof. Given any model M with a distinguished independent subset $H(M)$ of generics, we can always find an elementary extension N of M and a set $H(N)$ extending $H(M)$ such that for every generic 1-type $p(x, \mathrm{acl}(\vec{m}))$ (i.e. $SU(p(x)) = \omega$), where $\vec{m} \in M$, there is $d \in N$ such that $d \models p(x, \mathrm{acl}(\vec{m}))$ and $d \mathop{\smile\!\!\!\!\!\mid}_{H(M)} \vec{m}$. Add a similar statement for the extension property. Now apply a chain argument. □

In particular, for a SU-rank ω theory T, H-structures exist.

Lemma 2.7. *Let (M, H) and (N, H) be sufficiently saturated H-structures, $\vec{a} \in M$ and $\vec{a}' \in N$ H-independent tuples such that* $\text{tp}(\vec{a}, H(\vec{a})) = \text{tp}(\vec{a}', H(\vec{a}'))$. *Then* $tp_H(\vec{a}) = \text{tp}_H(\vec{a}')$.

Proof. Write $\vec{a} = \vec{a}_0\vec{a}_1\vec{h}$, where $\vec{a}_0$ is independent over $H(M)$, $\vec{h} = H(\vec{a}) \in H(M)$ and $\vec{a}_1 \in \mathcal{L}(\vec{a}_0\vec{h})$. Similarly write $\vec{a}' = \vec{a}_0'\vec{a}_1'\vec{h}'$.

It suffices to show that for any $b \in M$ there are $\vec{h}_1 \in H(M)$, $\vec{h}_1' \in H(N)$ and $b' \in N$ such that $\vec{a}\vec{h}_1 b$ and $\vec{a}'\vec{h}_1'b'$ are each H-independent, $\text{tp}(\vec{a}_0\vec{a}_1\vec{h}\vec{h}_1 b) = \text{tp}(\vec{a}_0'\vec{a}_1'\vec{h}'\vec{h}_1'b')$, and $b \in H(M)$ iff $b' \in H(N)$.

Case 1: $b \in \text{cl}(\vec{a}) \cap H(M)$. By H-independence of $\vec{a}$, we must have $b \in \text{cl}(\vec{h})$ and since H forms an independent set we must have $b \in \vec{h}$. Let $b' \in \vec{h}'$ be such that $\text{tp}(b'\vec{a}') = \text{tp}(b\vec{a})$ and the result follows. Here we can take $\vec{h}_1$ and $\vec{h}_1'$ to be empty

Case 2: $b \in H(M)$ and is non small over $\vec{a}$. Then $tp(b/\vec{a})$ is generic. By the density property, we can find $b' \in H(N)$ such that $\text{tp}(b'\vec{a}') = \text{tp}(b\vec{a})$. Here again we can take $\vec{h}_1$ and $\vec{h}_1'$ to be empty.

Case 3: $b \in \text{cl}(\vec{a})$. We claim that $b \mathop{\smile\!\!\!\!\!\mid}_{\vec{a}} H(M)$. Indeed let $\vec{h}_1$ (say of length k) in $H(M) \setminus \vec{h}$. Since $\vec{a}$ is H-independent, the elements in $H(M) \setminus \vec{h}$ are independent over $\vec{a}$ and thus $SU(\vec{h}_1/\vec{a}) = SU(\vec{h}_1/\vec{h}) = \omega k$. On the other hand $SU(b/\vec{a}) < \omega$, so the types $\text{tp}(b/\vec{a})$, $\text{tp}(\vec{h}_1/\vec{a})$ are orthogonal and the claim follows.

Thus the tuple $\vec{a}b$ is H-independent. Let $p(x, \vec{a}) = \text{tp}(b/\vec{a})$. Now use the extension property to find $b' \in N'$ such that $b' \models p(x, \vec{a}')$, $b' \mathop{\smile\!\!\!\!\!\mid}_{\vec{a}'} H(N)$, so by transitivity $\vec{a}'b'$ is H-independent.

Case 4: $b \in \text{cl}(H(M)\vec{a})$. Add a tuple $\vec{h}_1 \in H(M)$ such that $\vec{a}b\vec{h}_1$ is H-independent, and use Case 2 and Case 3.

Case 5: $b \notin \text{cl}(H(M)\vec{a})$. By the extension property, there is $b' \in N$ such that $b' \notin \text{cl}(H(N)\vec{a}')$ and $\text{tp}(b'\vec{a}') = \text{tp}(b\vec{a})$. The tuples stay H-independent, so again we can take $\vec{h}_1$ and $\vec{h}_1'$ to be empty.

□

The previous result has the following consequence:

Corollary 2.8. *All H-structures are elementarily equivalent.*

We write T^{ind} for the common complete theory of all H-structures of models of T.

Definition 2.9. We say that T^{ind} is *first order* if the $|T|^+$-saturated models of T^{ind} are again H-structures.

To axiomatize T^{ind} and to show that T^{ind} is first oder, we follow the ideas of [15, Prop 2.15], [3] and [2]. Here we use for the first time that T eliminates $\exists^{large}$. Recall that whenever T eliminates $\exists^{large}$ the expression *the formula $\varphi(x, \vec{b})$ is large* is first order.

We also need the following definition from [2, Definition 2.4]:

Definition 2.10. Let $\psi(\vec{y}, \vec{z})$ and $\varphi(\vec{x}, \vec{y})$ be $\mathcal{L}$-formulas. $Q_{\varphi,\psi}$ is the predicate which is defined to hold of a tuple $\vec{c}$ (in M) if for all $\vec{b}$ satisfying $\psi(\vec{y}, \vec{c})$, the formula $\varphi(\vec{x}, \vec{b})$ does not divide over $\vec{c}$.

The following result follows word by word from the proof of [2, Proposition 4.5], changing the elementary substructure for the predicate H:

Proposition 2.11. *The following are equivalent:*

1. $Q_{\varphi,\psi}$ *is type-definable (in* M *) for all* $\mathcal{L}$*-formulas* $\varphi(\vec{x},\vec{y}),\psi(\vec{y},\vec{z})$.
2. *The extension property is first order.*
3. *Any* $|T|^+$*-saturated model of* T^{ind} *satisfies the extension property.*

Corollary 2.12. *Let* T *be a simple theory of* SU*-rank omega that satisfies wnfcp. Then the extension property is first order.*

Proposition 2.13. *Assume* T *eliminates* $\exists^{large}$ *and that the predicates* $Q_{\varphi,\psi}$ *are* $\mathcal{L}$*-type-definable for all* $\mathcal{L}$*-formulas* $\varphi(\vec{x},\vec{y}),\psi(\vec{y},\vec{z})$*. Then* T^{ind} *is first order.*

Proof. The theory T^{ind} is described by T', the density property and the extension property.

T' is a first order property.

The density property can be described in first order by the scheme:

For all $\mathcal{L}$-formulas $\varphi(x,\vec{y})$

$\forall\vec{y}(\varphi(x,\vec{y})$ large $\implies \exists x(\varphi(x,\vec{y})\wedge x\in H))$.

Thus all saturated models of the scheme satisfy the density property. Finally by Proposition 2.11 any $|T|^+$-saturated model of T^{ind} satisfy the extension property. □

Notation 2.14. *Let* $(M,H(M))$ *be an* H*-structure and let* $A\subset M$*. We write* $\mathrm{cl}_H(A)$ *for* $\mathrm{cl}(AH(M))$ *and we call it the* small closure of A over H*.*

3. EXAMPLES

In this section we give a list of examples of simple theories of SU-rank ω that eliminate $\exists^{large}$ and where the extension property is first order. We also list some examples that eliminate the quantifier $\exists^{large}$ but where it remains as an open question if the extension property is first order.

3.1. **Differentially closed fields.**

Let $T = DCF_0$, the theory of differentially closed fields. This theory is stable of U rank ω and also $RM(DCF_0)=\omega$.

Let $p(x)$ be the unique generic type of the theory. This type is complete, stationary and definable over $\emptyset$. Let $\varphi(x,\vec{y})$ be a formula and let $\psi(\vec{y})$ be its p-definition. Then for $(K,d)\models DCF_0$, $\vec{a}\in K$, the formula $\varphi(x,\vec{a})$ is large iff $\psi(\vec{a})$. Thus this theory eliminates the quantifier $\exists^{large}$.

Now let us study the extension property. Recall that DCF_0 has quantifier elimination [12, Theorem 2.4] and eliminates imaginaries [12, Theorem 3.7]. It is proved in [12, Theorem 2.13] that DCF_0 has uniform bounding (i.e. it eliminates $\exists^\infty$) and thus it has nfcp. This is also explicitly explained in [12, page 52]. It follows by Corollary 2.12 that the extension property is first order.

3.2. **Free pseudoplane-infinite branching tree.**

Let T be the theory of the free pseudoplane, that is, a graph without cycles such that every vertex has infinitely many edges. The theory of the free pseudoplane is stable of U-rank ω and $MR(T)=\omega$. For every A, $\mathrm{acl}(A)=\mathrm{dcl}(A)=A\cup\{x|$ there are points $a,b\in A$ and a path connecting them passing trough $x\}$. For A algebraically closed and a a single element, $U(a/A)=d(a,A)$ where $d(a,A)$ is the minimum length of a path from a to an element of A or ω if there is no path; in this last case we say that a is at *infinite distance to* A or that a is *not connected to* A. Note that there is a unique generic type over A, namely the type of an element which is not connected to A.

The generic type is definable over $\emptyset$ and thus by definability of types T eliminates the quantifier $\exists^{large}$.

An H-structure (M, H) associated to T is an infinite collection of trees with an infinite collection of selected points $H(M)$ at infinite distance one from the other and with infinite many trees not connected to them. If $(N, H) \models Th(M, H)$, then N has infinitely many selected points $H(N)$ at infinite distance one from the other.

If (N, H) is $\aleph_0$-saturated, then by saturation it also has infinitely many trees which are not connected to the points $H(N)$. We will prove that in this case (N, H) is an H-structure. The density property is clear. Now let $A \subset N$ be finite and assume that $A = \operatorname{dcl}(A)$ and let $c \in N$. If $U(c/A) = \omega$ choose a point b in a tree not connected to $A \cup H$, then $\operatorname{tp}(c/A) = \operatorname{tp}(b/A)$ and $b \downarrow_A H$. If $U(c/A) = 0$ there is nothing to prove. If $U(c/A) = n > 0$, let a be the nearest point from A to c. Since there is at most one point of H connected to a and the trees are infinitely branching, we can choose a point b with $d(b, a) = n$ and such that $d(b, A \cup H) = n$; then $\operatorname{tp}(c/A) = \operatorname{tp}(b/A)$ and $b \downarrow_A H$. This proves that (N, H) is an H-structure and that that T^{ind} is first order.

3.3. **Vector space with a generic automorphism.** Let T be the theory of (infinite-dimensional) vector spaces over a division ring F, and let T_σ by its (unique) generic automorphism expansion.

This theory has a unique generic, which is definable over $\emptyset$. By definability of types, T_σ eliminates the quantifier $\exists^{large}$.

Now we prove that the extension property is first order.

Let (M, H) be an H-structure associated to T_σ, let $(N, H) \models Th(M, H)$ be $|T|^+$-saturated and let $a, \vec{b} \in N$.

Note that the type of the element a over a tuple $\vec{b}$ in T_σ is determined by

$$qftp^-(\sigma^{\mathbb{Z}}(a)/\sigma^{\mathbb{Z}}(\vec{b})),$$

where the superscript $-$ refers to the language of T, and

$$\sigma^{\mathbb{Z}}(\vec{c}) = \ldots, \sigma^{-1}(\vec{c}), \vec{c}, \sigma(\vec{c}), \sigma^2(\vec{c}), \ldots.$$

There are three possible situations for $\operatorname{tp}(a/\vec{b})$:

(1) $a \in span(\sigma^{\mathbb{Z}}(\vec{b}))$
(2) $a, \sigma(a), \ldots, \sigma^{n-1}(a)$ are independent over $\sigma^{\mathbb{Z}}(\vec{b})$, but

$$\sigma^n(a) \in span(a, \sigma(a), \ldots, \sigma^{n-1}(a)\sigma^{\mathbb{Z}}(\vec{b}))$$

(3) $\sigma^{\mathbb{Z}}(a)$ is independent over $\sigma^{\mathbb{Z}}(\vec{b})$

For the first case, we have that $a \in \operatorname{dcl}(\vec{b})$ and thus $a \downarrow_{\vec{b}} H$.

For the second case, assume now that $\sigma^n(a) \in span(a, \sigma(a), \ldots, \sigma^{n-1}(a)\sigma^{\mathbb{Z}}(\vec{b}))$. Since M is an $\aleph_0$-saturated, we can find $a', \vec{b}' \in M$ such that $\operatorname{tp}(a, \vec{b}) = \operatorname{tp}(a', \vec{b}')$ and since (M, H) is an H-structure we may assume that $a' \downarrow_{\vec{b}'} H$. In particular, the elements $a', \sigma(a'), \ldots, \sigma^{n-1}(a')$ do not satisfy any nontrivial linear combination with elements in $dcl(\vec{b}' H(M))$. Since $(N, H) \models Th((M, H))$ is $|T|^+$-saturated, we can find $(a'', \vec{b}) \models \operatorname{tp}(a', \vec{b}')$ such that $a'', \sigma(a''), \ldots, \sigma^{n-1}(a'')$ do not satisfy any nontrivial linear combination with elements in $dcl(\vec{b}H(N))$. This shows that $a'' \downarrow_{\vec{b}} H$ as we wanted.

For the third case, since $(N,H) \models (M,H)$, we have that $H(N)$ is an infinite collection of independent generics. Let $a_0, \dots, a_{n^2-1} \in H(N)$ be distinct and consider $c_0 = a_0 + \dots + a_{n-1}, \dots, c_{n-1} = a_{n^2-n} + \dots + a_{n^2-1}$. Then the elements $c_0, \dots, c_{n-1}$ are independent generics and neither one can be written as a linear combination of less that n elements in H. Since (N,H) is $|T|^+$-saturated, we can find infinitely many independent generics that are independent over $H(N)$. If $\sigma^{\mathbb{Z}}(a)$ is independent over $\sigma^{\mathbb{Z}}(\vec{b})$ we can choose a' generic independent from $\vec{b}H(N)$ and thus $a' \mathop{\smile\!\!\!\!\!\!\;|}_{\vec{b}} H$.

3.4. **Theories of Morley rank omega with definable Morley rank.** Let T be a ω-stable theory of rank ω and let $M \models T$ be $|T|^+$-saturated. Assume also that the Morley rank is definable, that is, for every formula $\varphi(x,\vec{y})$ without parameters and every $\alpha \in \{0,1,\dots,\omega\}$ there is a formula $\psi_\alpha(\vec{y})$ without parameters such that for $\vec{a} \in M$, $MR(\varphi(x,\vec{a})) \geq \alpha$ if and only if $\psi_\alpha(\vec{a})$. To simplify the notation, we will write $MR(\varphi(x,\vec{a})) \geq \alpha$ instead of $\psi_\alpha(\vec{a})$. We will prove that T^{ind} is first order.

Elimination of $\exists^{large}$. Consider first $\varphi(x,\vec{y})$ and let $\vec{b} \in M$. Then $\varphi(x,\vec{b})$ is large if and only if $MR(\varphi(x,\vec{b})) \geq \omega$, so T eliminates the quantifier $\exists^{large}$.

Extension property. Now assume that (M,H) is an H-structure and let $(N,H) \models Th(M,H)$ be $|T|^+$-saturated. Let $a \in N$ and let $\vec{b} \in N$. If $MR(\text{tp}(a/\vec{b})) = 0$ there is nothing to prove. Assume then that $MR(\text{tp}(a/\vec{b})) = n > 0$.

Let $\varphi(x,\vec{y}) \in \text{tp}(a,\vec{b})$ with $MR(\varphi(x,\vec{b})) = n$ and $Md(\varphi(x,\vec{b})) = Md(\text{tp}(a/\vec{b}))$. Let $(a',\vec{b}') \models tp(a,\vec{b})$ belong to M. Since (M,H) is an H-structure, we may assume that $a' \mathop{\smile\!\!\!\!\!\!\;|}_{\vec{b}'} H$ and thus for every formula $\theta(x,\vec{y},\vec{z})$ and every tuple $\vec{h} \in H$, if $MR(\theta(x,\vec{b}',\vec{h})) < MR(\varphi(x,\vec{b}')) = n$ then $\neg\theta(x,\vec{b}',\vec{h}) \in \text{tp}(a'/\vec{b}'H)$. So $(M,H) \models \forall d' MR(\varphi(x,\vec{d}')) \geq n \implies \exists c \varphi(c,\vec{d}') \wedge \forall \vec{h} \in H(MR(\theta(x,\vec{d}',\vec{h})) < n \implies \neg\theta(c,\vec{d}',\vec{h}))$.

Since $(N,H) \models TH(M,H)$ is $|T|^+$-saturated, we can find a' such that $MR(\varphi(a',\vec{b})) \geq n$ and whenever $\vec{h} \in H(N)$ and $\theta(x,\vec{b},\vec{h})$ is a formula with Morley rank smaller than n we have $\neg\theta(a',\vec{b},\vec{h})$. This shows that $MR(a'/\vec{b}H) = MR(a'/\vec{b}) = MR(a/\vec{b})$, $Md(a'/\vec{b}) = Md(a/\vec{b})$, both a and a' are generics of the formula $\varphi(x,\vec{b})$ and thus $\text{tp}(a/\vec{b}) = \text{tp}(a'/\vec{b})$. Finally by construction $a' \mathop{\smile\!\!\!\!\!\!\;|}_{\vec{b}} H$. It follows that T^{ind} is first order.

3.5. ***H*-triples.** Recall from [3] that if T_0 is supersimple SU-rank one theory whose pregeometry is not trivial, then $T = T_0^{ind}$ has SU-rank omega. The models of T are structures of the form (M,H_1), where $M \models T_0$ and H_1 is a acl_0-dense and acl_0-codense subset of M. We write $\mathcal{L}_0$ for the language associated to T_0 and $\mathcal{L}$ for the language associated to T. Similarly, we write acl_0 for the algebraic closure in the language $\mathcal{L}_0$ and for $A \subset M \models T_0$, we write $S_n^0(A)$ for the space of $\mathcal{L}_0$-n-types over A.

We will assume that T_0 has a strong form of non-triviality, namely for all $\mathcal{L}_0$-definable infinite sets $\varphi(x)$, there is an algebraic triangle inside $\varphi(x)$. So there is a set B and there are $a \models \varphi(x)$ and there are b,c with each of a,b,c acl_0-independent from B and such that $a \in \text{acl}_0(bcB) \setminus \text{acl}_0(bB)$. With this assumption, if $(M,H) \models T$, $A \subset M$ and $a \notin \text{acl}_0(AH_1)$, then $SU(\text{tp}(a/A)) = \omega$ and the generics in the sense of (M,H_1) have SU as required for the present paper.

In this subsection we change our notation and we let H_2 be a new predicate symbol that will be interpreted by a dense and codense T-generic subset of (M, H_1).

The structures (M, H_1, H_2) were already studied in [3]. We recall the definitions and the main result. The main tool for studying T^{ind} is to take into account the base theory T_0 and use triples.

Definition 3.1. We say that $(M, H_1(M), H_2(M))$ is an *H-triple* associated to T_0 if:

(1) $M \models T_0$, $H_1(M)$ is an acl_0-independent subset of M, $H_2(M)$ is an acl_0-independent subset of M over H_1.
(2) (Density property for H_1) If $A \subset M$ is finite dimensional and $q \in S_1^0(A)$ is non-algebraic, there is $a \in H_1(M)$ such that $a \models q$.
(3) (Density property for H_2/H_1) If $A \subset M$ is finite dimensional and $q \in S_1^0(A)$ is non-algebraic, there is $a \in H_2(M)$ such that $a \models q$ and $a \notin \mathrm{acl}_0(A \cup H_1(M))$.
(4) (Extension property) If $A \subset M$ is finite dimensional and $q \in S_1^0(A)$ is non-algebraic, there is $a \in M$, $a \models q$ and $a \notin \mathrm{acl}_0(A \cup H_1(M) \cup H_2(M))$.

It is observed in [3] that if $(M, H_1(M), H_2(M))$, $(N, H_1(N), H_2(N))$ are H-triples, then $Th(M, H_1(M), H_2(M)) = Th(N, H_1(N), H_2(N))$ and we denote the common theory by T_0^{tri}.

The folowing result is proved in [3] for geometric theories.

Proposition 3.2. *Let T be an SU rank one strongly non-trivial supersimple theory, let $M \models T$ and let $H_1(M) \subset M$, $H_2(M) \subset M$ be distinguished subsets. Then $(M, H_1(M), H_2(M))$ is a H_2-structure associated to T if and only if $(M, H_1(M), H_2(M))$ is an H-triple.*

Thus, to show that the class of H_2-structures associated to T is first order, it suffices to prove that this is the case for H-triples associated to T_0. As pointed out in [3] we have:

Proposition 3.3. *The theory T^{tri} is axiomatized by:*

(1) *T.*
(2) *$M \models T_0$, $H_1(M)$ is an acl_0-independent subset of M, $H_2(M)$ is an acl_0-independent subset of M over H_1.*
(3) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in H_1))$.
(4) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z} \forall \vec{y} \exists^{\leq n} x \psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in x)*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in H_2) \wedge$
$\forall w_1 \ldots \forall w_m \in H_1 \neg \psi(x, w_1, \ldots, w_m, \vec{y}))$
(5) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z} \forall \vec{y} \exists^{\leq n} x \psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in x)*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x \varphi(x, \vec{y}) \wedge$
$\forall w_1 \ldots \forall w_m \in H_1 \cup H_2 \neg \psi(x, w_1, \ldots, w_m, \vec{y}))$

Furthermore, if $(M, H, H_2) \models T^{tri}$ is $|T|^+$-saturated, then (M, H, H_2) is an H-triple.

Thus when T_0 is a strongly non-trivial supersimple SU-rank one theory, $T^{ind} = T^{tri}$ is first order.

3.6. H **structures of lovely pairs of** SU**-rank one theories.** Let T be a geometric theory, T_P its lovely pairs expansion, and let

$$\mathrm{cl}(-) = \mathrm{acl}(- \cup P(M))$$

be the small closure operator in a lovely pair (M, P). Our goal is to expand T_P to a theory T_P^{ind} in the language $\mathcal{L}_{PH} = \mathcal{L}_P \cup \{H\}$, by adding a cl-independent dense set to a model of T_P.

The following definition is analogous to Definition 3.1.

Definition 3.4. We say that an $\mathcal{L}_{PH}$-structure (M, P, H) is a PH-structure of T if

(1) $P(M)$ is an elementary substructure of M;
(2) $H(M)$ is acl-independent over $P(M)$;
(3) for any non-algebraic type $q \in S_1^T(A)$ over a finite-dimensional set $A \subset M$, q is realized in
 (density of P over H) $P(M) \setminus \mathrm{acl}(H(M)A)$;
 (density of H over P) $H(M) \setminus \mathrm{acl}(P(M)A)$;
 (extension) $M \setminus \mathrm{acl}(P(M)H(M)A)$.

Remark 3.5. *(a) It suffices to require $P(M)$ to be dense in the usual sense, i.e. q having a realization in $P(M)$.*

(b) We can get a PH-structure from an H-triple (M, H_1, H_2) (see previous example), by letting $P(M) = \mathrm{acl}(H_1)$.

(c) A usual elementary chain argument shows that any L_{PH} structure (M, P, H) satisfying (1,2) embeds in a PH-structure (N, P, H) so that $H(N) \downarrow_{H(M)} MP(N)$ and $P(N) \downarrow_{P(M)} MH(N)$. In particular, PH-structures exist.

(d) Reducts (M, P) and (M, H) of (M, P, H) are lovely pairs and H-structures, respectively.

While in linear examples the SU-rank of T_P is two instead of ω, the machinery for this paper still goes through we our current assumptions for cl.

Definition 3.6. We say that (M, P, H) is an cl-structure if

(1) (M, P) is a lovely pair and H is an cl-independent set
(2) (Density/coheir property for cl) If $A \subset M$ is finite dimensional and $q \in S_1^P(A)$ is large, there is $a \in H(M)$ such that $a \models q$.
(3) (Extension property) If $A \subset M$ is finite dimensional and $q \in S_1^P(A)$ is large, there is $a \in M$, $a \models q$ and $a \notin \mathrm{cl}(A \cup H(M))$.

Proposition 3.7. *(M, P, H) is an* cl*-structure if and only if (M, P, H) is a PH-structure.*

Proof. Assume first that (M, P, H) is a cl-structure. Then the pair (M, P) is lovely and thus (M, P, H) satisfies the density axiom for P. Now let $A \subset M$ be finite dimensional and let $q \in S_1(A)$ be non-algebraic. Let $\hat{q} \in S_1^P(A)$ be an extension of q that contains no small formula with parameters in A. Then by the Density/coheir property for cl it follows that there is $a \in H(M)$ such that $a \models \hat{q}$. In particular, $a \models q$ and $a \notin \mathrm{cl}(A)$ and thus we get the density property for H over P. Finally, since the same $\hat{q}$ is not small, there is $c \in M$, $c \models \hat{q}$ and $c \notin \text{cl}(A \cup H(M)) = \text{acl}(A \cup P(M) \cup H(M))$. Thus the extension property holds as well.

Now assume that (M, P, H) is an PH-structure. Then H is an cl-independent set, and by the density property for P and the extension property it follows that (M, P) is a lovely pair. Now let $A \subset M$ be finite dimensional and let $\hat{q} \in S_1^P(A)$ be non-small. We may enlarge A and assume that A is P-independent. Let q be the restriction of $\hat{q}$ to the language $\mathcal{L}$. Note that $\hat{q}$ is the unique extension of q to a non-small type. By the density for H over P, there is $a \in H(M)$ such that $a \models q$, $a \notin \text{cl}(A)$ and thus $a \models \hat{q}$. Finally the extension property follows from the extension property for PH-structures. $\square$

We will now show that the class of PH-structures is "first order", that is, that there is a set of axioms whose $|T|^+$-saturated models are the PH-structures. The axiomatization works as in H-triples.

Proposition 3.8. *Assume T eliminates $\exists^\infty$. Then the theory T_{PH} is axiomatized by:*

(1) T
(2) *axioms saying that P distinguishes an elementary substructure.*
(3) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in P))$.
(4) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z} \forall \vec{y} \exists^{\leq n} x \psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in x)*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in H) \wedge$
$\forall w_1 \ldots \forall w_m \in P \neg \psi(x, w_1, \ldots, w_m, \vec{y}))$
(5) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z} \forall \vec{y} \exists^{\leq n} x \psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in x)*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \notin P \wedge x \notin H) \wedge$
$\forall w_1 \ldots \forall w_m \in P \cup H \neg \psi(x, w_1, \ldots, w_m, \vec{y}))$
Furthermore, if $(M, P, H) \models T_{PH}$ is $|T|^+$-saturated, then (M, P, H) is a PH-structure.

Now we list a family of structures of SU-rank ω where we do know if the corresponding theory of H-structures is axiomatizable. In both cases it is open whether or not the extension property is first order.

3.7. **ACFA.** Let $T = ACFA$, (a completion) of the theory of algebraically closed fields with a generic automorphism. This theory is simple of SU rank ω and it is unstable.

Let $p(x)$ be the generic type of the theory, namely the type of a transformally independent element. This type is complete, stationary and definable over $\emptyset$. Let $\varphi(x, \vec{y})$ be a formula and let $\psi(\vec{y})$ be its p-definition. Then for $(K, \sigma) \models ACFA$, $\vec{a} \in K$, the formula $\varphi(x, \vec{a})$ is large iff $\psi(\vec{a})$. Thus this theory eliminates the quantifier $\exists^{large}$.

Question Does the extension property hold for ACFA? Does T_0 satisfy wnfcp?

3.8. **Hrushovski amalgamation without collapsing.** In this subsection we follow the presentation of Hrushovski amalgamations from [16], all the results we mention can be found in [16]. Let $\mathcal{L} = \{R\}$ where R stands for a ternary relation. We let $\mathcal{C}$ be the class of $\mathcal{L}$-structures where R is symmetric and not reflexive. For $A \in \mathcal{C}$ a finite structure we let $\delta(A) = |A| - |R(A)|$ and we let $\mathcal{C}^0_{fin}$ be the subclass of $\mathcal{C}$ consisting of all finite $\mathcal{L}$-structures M where for $A \subset M$ we have $\delta(A) \geq 0$. Finally M^0 stands for the Fraïssé limit of the class $\mathcal{C}^0_{fin}$. Let T_0 be the theory of M^0, then $MR(T_0) = \omega$ and $Md(T_0) = 1$.

Now let $M \models T_0$ and for $A \subset M$ finite we define $d(A) = \inf\{\delta(B) : A \subset B\}$. Then d is the dimension function of a pregeometry and that for an element a and a set B, $d(a/B) = 1$ if and only if $MR(a/B) = \omega$ if and only if $U(a/B) = \omega$. Thus the pregeometry studied in [16] corresponds to the pregeometry associated to cl. Since the theory T_0 has a unique generic type, by definability of types the theory T_0 eliminates the quantifier $\exists^{large}$.

Question Does the extension property hold for T_0? Does T_0 satisfy nfcp?

4. Definable sets in H-structures

Fix T a SUrank ω theory and let $(M, H(M)) \models T^{ind}$. Our next goal is to obtain a description of definable subsets of M and $H(M)$ in the language $\mathcal{L}_H$.

Notation 4.1. *Let $(M, H(M))$ be an H-structure. Let $\vec{a}$ be a tuple in M. We denote by $\mathrm{etp}_H(\vec{a})$ the collection of formulas of the form $\exists x_1 \in H \dots \exists x_m \in H \varphi(\vec{x}, \vec{y})$, where $\varphi(\vec{x}, \vec{y})$ is an $\mathcal{L}$-formula such that there exists $\vec{h} \in H$ with $M \models \varphi(\vec{h}, \vec{a})$.*

Lemma 4.2. *Let $(M, H(M))$, $(N, H(N))$ be H-structures. Let $\vec{a}$, $\vec{b}$ be tuples of the same arity from M, N respectively. Then the following are equivalent:*

(1) $\mathrm{etp}_H(\vec{a}) = \mathrm{etp}_H(\vec{b})$.
(2) *$\vec{a}$, $\vec{b}$ have the same $\mathcal{L}_H$-type.*

Proof. Clearly (2) implies (1). Assume (1), then $\mathrm{tp}(\vec{a}) = \mathrm{tp}(\vec{b})$.

Claim $\dim_{\mathrm{cl}}(\vec{b}/H) = \dim_{\mathrm{cl}}(\vec{a}/H)$.

Let $\vec{h} = (h_1, \dots, h_l) \in H(M)$ be such $k := \dim_{\mathrm{cl}}(\vec{a}/\vec{h}) = \dim_{\mathrm{cl}}(\vec{a}/H(M))$. We may assume that $\vec{a}^1 = (a_1, \dots, a_k)$ are independent over H and $\vec{a}^2 = (a_{k+1}, \dots, a_n) \in \mathrm{cl}(a_1, \dots, a_k, h_1, \dots, h_l)$. Choose $\psi(\vec{x}, \vec{y}, \vec{z})$ such that for any $\vec{b} \in M$, $\vec{c} \in M$ $\psi(\vec{b}, \vec{c}, \vec{z})$ is always small in $\vec{z}$ and $M \models \psi(\vec{h}, \vec{a}^1, \vec{a}^2)$. Since $\mathrm{etp}_H(\vec{a}) = \mathrm{etp}_H(\vec{b})$ we get that $dim_{\mathrm{cl}}(\vec{b}/H) \leq k$. A similar argument shows that $\dim_{\mathrm{cl}}(\vec{a}/H(M)) \leq \dim_{\mathrm{cl}}(\vec{b}/H(N))$.

Claim $\mathrm{tp}_H(\vec{b}) = \mathrm{tp}_H(\vec{a})$.

As before, let $\vec{h} = (h_1, \dots, h_l) \in H(M)$ be such that $k := \dim_{\mathrm{cl}}(\vec{a}/\vec{h}) = dim_{\mathrm{cl}}(\vec{a}/H(M))$. Then $\vec{a}\vec{h}$ is H-independent. Since N is saturated as an $\mathcal{L}$-structure there are $\vec{h}' = (h'_1, \dots, h'_l) \in H$ such that $\mathrm{tp}(\vec{a}, \vec{h}) = \mathrm{tp}(\vec{b}, \vec{h}')$. By the claim above $\vec{b}\vec{h}'$ is H-independent, so the result follows from Lemma 2.7. $\square$

Now we are interested in the $\mathcal{L}_H$-definable subsets of $H(M)$. This material is very similar to the results presented in [5].

Lemma 4.3. *Let $(M_0, H(M_0)) \preceq (M_1, H(M_1))$ and assume that $(M_1, H(M_1))$ is $|M_0|$-saturated. The M_0 (seen as a subset of M_1) is a H-independent set.*

Proof. Assume not. Then there are $a_1, \dots, a_n \in M_0 \setminus H(M_0)$ such that $a_n \in \mathrm{cl}(a_1, \dots, a_{n-1}, H(M_1))$ and $a_n \notin \mathrm{cl}(a_1, \dots, a_{n-1}, H(M_0))$. Let $\varphi(x, \vec{y}, \vec{z})$ be a formula which is always small on x and $\vec{b} \in H(M_1)_{\vec{z}}$ be a tuple such that $\varphi(a_n, a_1, \dots, a_{n-1}, \vec{b})$

holds. Since $(M_0, H(M_0)) \preceq (M_1, H(M_1))$ there is $\vec{b}' \in H(M_0)_{\vec{z}}$ such that

$$\varphi(a_n, a_1, \ldots, a_{n-1}, \vec{b}') \wedge \neg\exists^{large} x\varphi(x, a_1, \ldots, a_{n-1}, \vec{b}')$$

holds, so $a_n \in \operatorname{acl}(a_1, \ldots, a_{n-1}, H(M_0))$, a contradiction. $\square$

Proposition 4.4. *Let $(M, H(M))$ be an H-structure and let $Y \subset H(M)^n$ be $\mathcal{L}_H$-definable. Then there is $X \subset M^n$ $\mathcal{L}$-definable such that $Y = X \cap H(M)^n$.*

Proof. Let $(M_1, H(M_1)) \succeq (M, H(M))$ be κ-saturated where $\kappa > |M| + |L|$ and let $\vec{a}, \vec{b} \in H(M_1)^n$ be such that $\operatorname{tp}(\vec{a}/M) = \operatorname{tp}(\vec{b}/M)$. We will prove that $\operatorname{tp}_H(\vec{a}/M) = \operatorname{tp}_H(\vec{b}/M)$ and the result will follow by compactness. Since $\vec{a}, \vec{b} \in H(M_1)^n$, we get by Lemma 4.3 that $M\vec{a}$, $M\vec{b}$ are H-independent sets and thus by Lemma 2.7 we get $\operatorname{tp}_H(\vec{a}/M) = \operatorname{tp}_H(\vec{b}/M)$. $\square$

Definition 4.5. Let $(M, H) \models T^{ind}$ be saturated. We say that an $\mathcal{L}_H$-formula $\psi(x, \vec{c})$ defines a *H-large* subset of M if there is $b \models \psi(x, \vec{c})$ such that $b \notin \operatorname{cl}(H\vec{c})$. This is equivalent as requiring that there are infinitely many realizations of $\psi(x, \vec{c})$ that are not small over $H(M)\vec{c}$.

Definition 4.6. Let $(M, H) \models T^{ind}$ be κ-saturated and let $A \subset M$ be smaller than κ. Let $\vec{b} \in M$ be a tuple. We say that $\vec{b}$ is in the *H-small closure* of A if $\vec{b} \in \operatorname{cl}(AH(M))$. Let $X \subset M^n$ be A-definable. We say that X is *H-small* if $X \subset \operatorname{cl}(A \cup H(M))$.

Proposition 4.7. *Let $(M, H(M))$ be an H-structure. Let $\vec{a} = (a_1, \ldots, a_n) \in M$. Then there is a unique smallest tuple $\vec{h} \in H(M)$ such that $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{\vec{h}} H$.*

Proof. Since T is supersimple, there is a finite tuple $\vec{h} \in H$ such that $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{\vec{h}} H$. Choose such a tuple so that $|\vec{h}|$ (the length of the tuple) is minimal. We will now show such a tuple $\vec{h}$ is unique (up to permutation).

We can write $\vec{a} = (\vec{a}_1, \vec{a}_2)$ so that $\vec{a}_1$ is independent tuple of generics which is independent from $H(M)$ and $\vec{a}_2 \in \operatorname{cl}(\vec{a}_1)$. If $\vec{a}_2 = \emptyset$, then $\vec{h} = \emptyset$ and the result follows. So we may assume that $\vec{a}_2 \neq \emptyset$.

Then $\vec{a}_2 \in \operatorname{cl}(\vec{a}_1, \vec{h})$. Let $\vec{h}'$ be another such tuple. Let $\vec{h}_1$ be the list of common elements in both $\vec{h}$ and $\vec{h}'$, so we can write $\vec{h} = (\vec{h}_1, \vec{h}_2)$ and $\vec{h}' = (\vec{h}_1, \vec{h}'_2)$.

Claim $\vec{h}_2 = \vec{h}'_2 = \emptyset$.

Assume otherwise. Then there is $c \in \vec{a}_2$ such that $c \in \operatorname{cl}(\vec{a}_1, \vec{h}_1, \vec{h}_2) \setminus \operatorname{cl}(\vec{a}_1, \vec{h}_1)$. Since $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{\vec{h}'} H$, we must have that $c \in \operatorname{cl}(\vec{a}_1, \vec{h}_1, \vec{h}'_2) \setminus \operatorname{cl}(\vec{a}_1, \vec{h}_1)$. By the exchange property $\dim_{\mathrm{cl}}(\vec{h}'_2/\vec{a}_1\vec{h}_1\vec{h}_2) < \dim_{\mathrm{cl}}(\vec{h}'_2/\vec{a}_1\vec{h}_1)$. Since $\vec{a}_1$ is a tuple of generic elements that are independent over H we get that $\dim_{\mathrm{cl}}(\vec{h}'_2/\vec{h}_1\vec{h}_2) < \dim_{\mathrm{cl}}(\vec{h}'_2/\vec{h}_1)$ and since H is independent, $\vec{h}_2$ has a common element with $\vec{h}'_2$, a contradiction. $\square$

Remark 4.8. *Let $(M, H(M))$ be an H-structure. Let $\vec{a} = (a_1, \ldots, a_n) \in M$ and let $C \subset M$ be such that C is H-independent. As before, there is a unique smallest tuple $\vec{h} \in H(M)$ such that $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{\vec{h}C} H$.*

Notation 4.9. *Let $(M, H(M))$ be an H-structure. Let $\vec{a} = (a_1, \ldots, a_n) \in M$. Let $\vec{h} \in H(M)$ be the smallest tuple such that $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{\vec{h}} H$. We call $\vec{h}$ the H-*basis *of $\vec{a}$ and we denote it as $HB(\vec{a})$. Given $C \subset M$ such that C is H-independent, let $\vec{h} \in H(M)$ the smallest tuple such that $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{C\vec{h}} H$. We call $\vec{h}$ the H-*basis of $\vec{a}$ over C *and we* *denote it as $HB(\vec{a}/C)$. Note that H-basis is unique up to permutation, therefore we will view the H-basis $\vec{h} = (h_1, \ldots, h_k)$ either as a finite set $\{h_1, \ldots, h_k\}$ or as the imaginary representing this finite set. If we view it as a tuple, we will explicitly say so.*

Proposition 4.10. *Let $(M, H(M))$ be an H-structure. Let $a_1, \ldots, a_n, a_{n+1} \in M$ and let $C \subset M$ be such that C is H-independent. Then $HB(a_1, \ldots, a_n, a_{n+1}/C) = HB(a_1, \ldots, a_n/C) \cup HB(a_{n+1}/Ca_1, \ldots, a_n HB(a_1, \ldots, a_n/C))$, where all H-basis are seen as sets.*

Proof. Let $\vec{h}_1 = HB(a_1, \ldots, a_n/C)$. First note that since $a_1, \ldots, a_n \downarrow_{C\vec{h}_1} H$, then the set $a_1, \ldots, a_n C\vec{h}_1$ is H-independent and we can define $\vec{h}_2 = HB(a_{n+1}/Ca_1, \ldots, a_n\vec{h}_1)$. Finally, let $\vec{h} = HB(a_1, \ldots, a_n, a_{n+1}/C)$.

Claim $\vec{h} \subset \vec{h}_1\vec{h}_2$.

We have $a_1, \ldots, a_n \downarrow_{C\vec{h}_1} H$ and $a_{n+1} \downarrow_{C\vec{h}_1\vec{h}_2 a_1,\ldots,a_n} H$, so by transitivity, $a_1, \ldots, a_n a_{n+1} \downarrow_{C\vec{h}_1\vec{h}_2} H$ and by the minimality of an H-basis, we have $\vec{h} \subset \vec{h}_1\vec{h}_2$.

Claim $\vec{h} \supset \vec{h}_1\vec{h}_2$.

By definition, $a_1, \ldots, a_n a_{n+1} \downarrow_{C\vec{h}} H$, so $a_1, \ldots, a_n \downarrow_{C\vec{h}} H$ and by minimality we have $\vec{h}_1 \subset \vec{h}$. We also get by transitivity that $a_{n+1} \downarrow_{Ca_1,\ldots,a_n\vec{h}_1\vec{h}} H$ and by the minimality of H-basis we get $\vec{h}_2 \subset \vec{h}$ as desired. $\square$

Proposition 4.11. *Let $(M, H(M))$ be an H-structure. Let $a_1, \ldots, a_n \in M$ and let $C \subset D \subset M$ be such that C, D are H-independent. Assume that there is $h \in HB(a_1, \ldots, a_n/C) \setminus HB(a_1, \ldots, a_n/D)$. Then $h \in D$.*

Proof. Write $h_D = HB(a_1, \ldots, a_n/D)$ and see it as a set. Then $a_1, \ldots, a_n D \downarrow_{h_D H(D)} H$ and $a_1, \ldots, a_n \downarrow_{h_D C H(D)} H$. By minimality of $HB(a_1, \ldots, a_n/C)$ we get that $HB(a_1, \ldots, a_n/C) \subset h_D H(D)$ and thus if $h \in HB(a_1, \ldots, a_n/C) \setminus h_D$, we must have $h \in H(D)$. $\square$

We will now apply the H-basis to characterize definable sets in terms of $\mathcal{L}$-definable sets.

Proposition 4.12. *Let $(M, H(M))$ be an H-structure and let $Y \subset M$ be $\mathcal{L}_H$-definable. Then there is $X \subset M$ $\mathcal{L}$-definable such that $Y \triangle X$ is H-small, where $\triangle$ stands for a boolean connective for the symmetric difference.*

Proof. If Y is H-small or H-cosmall, the result is clear, so we may assume that both Y and $M \setminus Y$ are H-large. Assume that Y is definable over $\vec{a}$ and that $\vec{a} = \vec{a}HB(\vec{a})$. Let $b \in Y$ be such that $b \notin \mathrm{cl}(\vec{a}H)$ and let $c \in M \setminus Y$ be such that $c \notin \mathrm{cl}(\vec{a}H)$. Then $b\vec{a}$, $c\vec{a}$ are H-independent and thus there is X_{bc} an $\mathcal{L}$-definable set such that $b \in X_{bc}$ and $c \notin X_{bc}$. By compactness, we may get a single $\mathcal{L}$-definable set X such that for $b' \in Y$ and $c' \in M \setminus X$ not in the H-small closure of $\vec{a}$, we have $b' \in X$ and $c' \in M \setminus Z$. This shows that $Y \triangle X$ is H-small. $\square$

Our next goal is to characterize the algebraic closure in H-structures. The key tool is the following result:

Lemma 4.13. *Let $(M, H(M))$ be an H-structure, and let $A \subset M$ be* acl*-closed and H-independent. Then A is acl_H-closed.*

Proof. Suppose $a \in M$, $a \notin A$. If $a \notin \mathrm{cl}(AH)$, then $A \cup \{a\}$ is H-independent, and using the extension property, we can find a_i, $i \in \omega$, acl-independent over $A \cup H(M)$, realizing $\mathrm{tp}(a/A)$. By Lemma 2.7, each a_i realizes $\mathrm{tp}_H(a/A)$, and thus $a \notin \mathrm{acl}_H(A)$.

If $a \in \mathrm{cl}(AH)$, take a minimal tuple $\vec{h} \in H(M)$ such that $a \in \mathrm{acl}(A\vec{h})$. Using conjugates of $\vec{h}$ over A it is easy to see that $tp(a/A)$ has ∞-many realizations. $\square$

Corollary 4.14. *Let $(M, H(M))$ be an H-structure, and let $A \subset M$. Then* $\mathrm{acl}_H(A) = \mathrm{acl}(A, HB(A))$.

Proof. By Proposition 4.7, it is clear that $HB(A) \in \mathrm{acl}(A)$, so $\mathrm{acl}_H(A) \supset \mathrm{acl}(A, HB(A))$. On the other hand, $A \cup HB(A)$ is H-closed, so by the previous Proposition, $\mathrm{acl}(A \cup HB(A)) = \mathrm{acl}_H(A \cup HB(A))$ and thus $\mathrm{acl}_H(A) \subset \mathrm{acl}(A, HB(A))$ $\square$

5. Supersimplicity

In this section we prove that T^{ind} is supersimple and characterize forking in T^{ind}.

Theorem 5.1. *The theory T^{ind} is supersimple.*

Proof. We will prove that non-dividing has local character.

Let $(M, H(M)) \models T^{ind}$ be saturated. Let $C \subset D \subset M$ and assume that $C = \mathrm{acl}_H(C)$ and $D = \mathrm{acl}_H(D)$. Note that both C and D are H-independent. Let $\vec{a} \in M$. We will find a collection of conditions for the type of $\vec{a}$ over C that guarantee that $\mathrm{tp}_H(\vec{a}/D)$ does not divide over C.

We may write $\vec{a} = (\vec{a}_1, \vec{a}_2) \in M$ so that $\vec{a}_1$ is an independent tuple of generics over DH, $\vec{a}_2$ is a tuple such that $\vec{a}_2 \in \mathrm{cl}(\vec{a}_1 DH)$.

Assume that the following conditions hold for C:

(1) $HB(\vec{a}/D) = HB(\vec{a}/C)$.
(2) $SU(\vec{a}_2/C\vec{a}_1H) = SU(\vec{a}_2/D\vec{a}_1H)$

Claim $\mathrm{tp}_H(\vec{a}/D)$ does not divide over C.

Let $p(\vec{x}, D) = tp(\vec{a}_1, D)$. Let $\{D_i : i \in \omega\}$ be an $\mathcal{L}_H$-indiscernible sequence over C. Since $\vec{a}_1$ is an independent tuple of generics over D, $tp(\vec{a}_1/D)$ does not divide over C and $\cup_{i\in\omega} p(\vec{x}, D_i)$ is consistent. We can find $\vec{a}_1' \models \cup_{i\in\omega} p(\vec{x}, D_i)$ such that $\{\vec{a}_1' D_i : i \in \omega\}$ is indiscernible and $\vec{a}_1'$ is an independent tuple of generics over $\cup_{i\in\omega} D_i$. By the generalized extension property, we may assume that $\vec{a}_1'$ is independent over $\cup_{i\in\omega} D_i H$. Note that $\vec{a}_1 D$ is H-independent, $\vec{a}_1' D_i$ is also H-independent for any $i \in \omega$. So by Lemma 2.7 $\mathrm{tp}_H(\vec{a}_1 D) = \mathrm{tp}_H(\vec{a}_1' D_i)$ for any $i \in \omega$.

Now let $\vec{h} = HB(\vec{a}/C)$ (viewed as a tuple) and let $q(\vec{y}, \vec{a}_1, D) = tp(\vec{h}, \vec{a}_1, D)$. Note that $\vec{h}$ is an independent tuple of generics over $\vec{a}_1 D$ (as well as an independent tuple over $\vec{a}_1 C$). Since $\{D_i \vec{a}_1' : i \in \omega\}$ is an $\mathcal{L}$-indiscernible sequence, there is $\vec{h}' \models \cup_{i\in\omega} q(\vec{y}, \vec{a}_1', D_i)$. We may assume that $\vec{h}'$ is independent from $\cup_{i\in\omega} D_i \vec{a}_1'$ and thus it is a tuple of generics over $\cup_{i\in\omega} D_i \vec{a}_1'$. Furthermore we may assume that the sequence $\{D_i \vec{a}_1' : i \in \omega\}$ is indiscernible over $\vec{h}'$.

By the generalized coheir/density property, we may assume that $\vec{h}' \in H$. Note that since each $\vec{a}_1' D_i$ is H-independent, then $\vec{h}' \vec{a}_1' D_i$ is also H-independent. On the other hand, $\mathrm{tp}(\vec{h}, \vec{a}_1, D) = \mathrm{tp}(\vec{h}', \vec{a}_1', D_i)$ for each i, so by Lemma 2.7 we have $\mathrm{tp}_H(\vec{h}, \vec{a}_1, D) = \mathrm{tp}_H(\vec{h}', \vec{a}_1', D_i)$. This shows that $\mathrm{tp}(\vec{a}_1, \vec{h}/D)$ does not divide over C.

Now consider $t(\vec{z}, \vec{a}_1, \vec{h}, D) = tp(\vec{a}_2, \vec{a}_1, \vec{h}, D)$. By the assumption $SU(\vec{a}_2/C\vec{a}_1\vec{h}) = SU(\vec{a}_2/D\vec{a}_1\vec{h})$, so $tp(\vec{a}_2/\vec{a}_1\vec{h}D)$ does not divide over $\vec{a}_1\vec{h}C$. Since $tp(\vec{a}_1\vec{h}D) = tp(\vec{a}'_1\vec{h}'D_0)$, then $t(\vec{z}, \vec{a}'_1, \vec{h}', D_0)$ does not divide over $\vec{a}'_1\vec{h}'C$.

Since $\{D_i : i \in \omega\}$ is an $\mathcal{L}$-indiscernible sequence over $C\vec{a}'_1\vec{h}'$, there is $\vec{a}'_2 \models \cup_{i\in\omega} t(\vec{z}, \vec{a}'_1, \vec{h}', D_i)$. We may assume as before that $\vec{a}'_2 \downarrow_{\vec{a}'_1\vec{h}'C} \cup_i D_i$.

By the extension property, we may assume that $\vec{a}'_2 \downarrow_{\vec{a}'_1\vec{h}'\cup_i D_i} H$. Using transitivity we also have $\vec{a}'_2 \downarrow_{\vec{a}'_1\vec{h}'C} H \cup_i D_i$ and it follows that $\vec{a}'_2 \downarrow_{\vec{a}'_1\vec{h}'D_i} H$, so we have $\vec{a}'_1\vec{a}'_2 \downarrow_{\vec{h}'D_i} H$, and thus, also $\vec{a}'_1\vec{a}'_2 D_i \downarrow_{\vec{h}'H(D_i)} H$ for each index i.

Since both $\vec{a}_1\vec{a}_2\vec{h}D$, $\vec{a}'_1\vec{a}'_2\vec{h}'D_i$ are H-independent and $\mathrm{tp}(\vec{a}'_1\vec{a}'_2\vec{h}'D_i) = \mathrm{tp}(\vec{a}_1\vec{a}_2\vec{h}D)$ by Lemma 2.7 $\mathrm{tp}_H(\vec{a}'_1\vec{a}'_2\vec{h}'D_i) = \mathrm{tp}_H(\vec{a}_1\vec{a}_2\vec{h}D)$.

This shows that $\mathrm{tp}(\vec{a}/D)$ does not divide over C.

Since T is supersimple, for any D and $\vec{a}$ we can always choose a finite subset C_0 of D such that $C = \mathrm{acl}_H(C_0)$ satisfies the conditions (1) and (2) above. This shows that T^{ind} is supersimple. $\square$

Proposition 5.2. *Let $(M, H) \models T^{ind}$ be saturated, let $C \subset D \subset M$ be such that $C = \mathrm{acl}_H(C)$, $D = \mathrm{acl}_H(D)$ and let $a \in M$. Then $\mathrm{tp}(a/D)$ forks over C iff $a \in D \setminus C$ or $a \in \mathrm{cl}(HD) \setminus \mathrm{cl}(HC)$ or $HB(a/C) \supsetneq HB(a/D)$ or, $HB(a/C) = HB(a/D)$ and $SU(a/CH) \neq SU(a/DH)$.*

Proof. In the proof of Theorem 5.1 we showed that if $a \in C$ or if, $HB(a/C) = HB(a/D)$ and $SU(a/CH) = SU(a/DH)$ then $\mathrm{tp}(a/D)$ does not fork over C. So it remains to show the other direction, which we do case by case.

Case 1: Assume that $a \in D \setminus C$, then a becomes algebraic over D and $\mathrm{tp}(a/D)$ forks over C.

Case 2: Assume that $a \in \mathrm{cl}(DH) \setminus \mathrm{cl}(CH)$. Then $SU(tp(a/DH)) < \omega$ and $SU(tp(a/CH)) = \omega$. We will prove that $tp_H(a/D)$ divides over C.

Let $\vec{d} \in D$ and let $\vec{c} \in C$ be such that $a \in \mathrm{cl}(\vec{c}\vec{d}H)$, so $SU(tp(a/\vec{d}\vec{c}H)) < \omega$. By additivity of Lascar rank, we can choose $\vec{d}$ to be independent generics over HC. Let $\vec{h} \in H$ be such that $a \in \mathrm{cl}(\vec{c}\vec{d}\vec{h})$. Let $p(x, \vec{y}) = tp_H(a, \vec{d}/C)$.

Let $\{\vec{d}_i : i \in \omega\}$ be an $\mathcal{L}$ -indiscernible sequence in $\mathrm{tp}(\vec{d}/C)$ over C such that $\{\vec{d}_i : i \in \omega\}$ is independent over C. By the generalized extension property, we may assume that $\{\vec{d}_i : i \in \omega\}$ is independent over HC. Note that by Lemma 2.7 $\{\vec{d}_i : i \in \omega\}$ is an $\mathcal{L}_H$-indiscernible sequence of generics over C. Assume, in order to get a contradiction, that there is $a' \models \cup_{i\in\omega} p(x, \vec{d}_i)$. Then there are $\{\vec{h}_i : i \in \omega\}$ such that $a' \in \mathrm{cl}(\vec{d}_i, \vec{c}, \vec{h}_i)$ for every i, that is, $SU(a'/\vec{d}_i, \vec{c}, \vec{h}_i) < \omega$. But $a' \notin \mathrm{cl}(CH)$, so $\vec{d}_0 \not\downarrow_{CH} \vec{d}_1$, a contradiction.

Case 3: Assume that $HB(a/D) \neq HB(a/C)$ and $a \in \mathrm{cl}(CH)$. Then $HB(a/D)$ is a proper subset of $HB(a/C)$. Write $\vec{h}_C = HB(a/C)$, $\vec{h}_D = HB(a/D)$ and let $\vec{h}_E \in H$ be such that $\vec{h}_C = \vec{h}_D\vec{h}_E$. Note that $\vec{h}_E \neq \emptyset$ and that $\vec{h}_E$ is an independent tuple over C.

Let $p(x, \vec{y}) = tp_H(a, \vec{h}_E/C)$. Let $\{\vec{h}^i_E : i \in \omega\}$ be an $\mathcal{L}$-indiscernible sequence in $\mathrm{tp}(\vec{h}_E/C)$ such that $\{\vec{h}^i_E : i \in \omega\}$ is independent over C. Then by the generalized density property, we may assume that the sequence $\{\vec{h}^i_E : i \in \omega\}$ belongs to H. Note that by Lemma 2.7, the sequence $\{\vec{h}^i_E : i \in \omega\}$ is $\mathcal{L}_H$-indiscernible over C. We will show that $\cup_{i\in\omega} p(x, \vec{h}^i_E)$ is inconsistent.

Assume, not, so there is $a' \models \cup_{i\in\omega} p(x, \vec{h}^i_E)$. Then we can find $\vec{h}_{D_i}$ in H such that $HB(a'/C) = \vec{h}_{D_i}\vec{h}^i_E$. Since the $\vec{h}^i_E$ are independent, we get that the HB basis of a' over C is not unique, a contradiction.

Case 4: Assume that $a \in \mathrm{cl}(HC)$, that $HB(a/D) = HB(a/C)$ and $SU(a/CH) < SU(a/DH)$.

Let $\vec{h} = HB(a/D) = HB(a/C)$, so $SU(a/C\vec{h}) < SU(a/D\vec{h})$ and $a \not\perp_{C\vec{h}} D$. Write $\vec{h}_D = H(D) \setminus H(C)$. Note that $\vec{h}$ is independent over D. Let $p(x, \vec{h}, D) = tp(a, \vec{h}, D)$ and let $\{D_i : i \in \omega\}$ be an $\mathcal{L}$-Morley sequence in $\mathrm{tp}(D/C\vec{h})$ such that $\{p(x, \vec{h}, D_i) : i \in \omega\}$ is k-inconsistent. Let $\vec{h}_{D_i}$ be such that $tp(D, \vec{h}_D) = tp(D_i, \vec{h}_{D_i})$, then $\{\vec{h}_{D_i} : i \in \omega\}$ is a $\mathcal{L}$-Morley sequence in $\mathrm{tp}(\vec{h}_D/C\vec{h})$. By the density property, we may choose the elements in H. By the extension property, we can realize $tp((D_i : i \in \omega)/C \cup_{i\in\omega} \vec{h}_{D_i})$ independent from H over $C \cup_{i\in\omega} \vec{h}_{D_i}$. Then $\mathrm{tp}(D, \vec{h}_D/C\vec{h}) = \mathrm{tp}(D_i, \vec{h}_{D_i}/C\vec{h})$ so by Lemma 2.7, $\mathrm{tp}_H(D, \vec{h}_D/C\vec{h}) = \mathrm{tp}_H(D_i, \vec{h}_{D_i}/C\vec{h})$ so we get that $\{D_i : i \in \omega\}$ is a sequence in $\mathrm{tp}_H(D/C\vec{h})$ such that $\{p(x, \vec{h}, D_i) : i \in \omega\}$ is k-inconsistent. Using Erdö-Rado, we can change $\{D_i : i \in \omega\}$ for an indiscernible sequence in $\mathrm{tp}_H(D/C\vec{h})$ with the property that $\{p(x, \vec{h}, D_i) : i \in \omega\}$ is k-inconsistent. This proves that $\mathrm{tp}_H(a/D\vec{h})$ divides over $C\vec{h}$. But since $\vec{h} \in \mathrm{acl}_H(Ca)$ we also get that $\mathrm{tp}_H(a/D)$ divides over C.

□

Corollary 5.3. *Let $(M, H) \models T^{ind}$ be saturated, let $C \subset D \subset M$ be such that C and D are H-independent and let $a_1, \ldots, a_n \in M$. We may reorder the tuple and assume that there is $k \leq n$ such that $a_1, \ldots, a_k$ are independent generics over CH and $a_{k+1}, \ldots, a_n \in \mathrm{cl}(a_1, \ldots, a_k, C, H)$. Then $\mathrm{tp}_H(a_1, \ldots, a_n/D)$ forks over C iff*

1. *$dim_{\mathrm{cl}}(a_1, \ldots, a_n/\mathrm{cl}(HD)) < dim_{\mathrm{cl}}(a_1, \ldots, a_n/\mathrm{cl}(HD))$ or*
2. *$dim_{\mathrm{cl}}(a_1, \ldots, a_n/\mathrm{cl}(HD)) = dim_{\mathrm{cl}}(a_1, \ldots, a_n/\mathrm{cl}(HD))$ and $HB(a_1, \ldots, a_n/C) \supsetneq HB(a_1, \ldots, a_n/D)$ or,*
3. *$dim_{\mathrm{cl}}(a_1, \ldots, a_n/\mathrm{cl}(HD)) = dim_{\mathrm{cl}}(a_1, \ldots, a_n/\mathrm{cl}(HD))$, $HB(a_1, \ldots, a_n/C) = HB(a_1, \ldots, a_n/D)$ and*

$$SU(a_{k+1}, \ldots, a_n/a_1, \ldots, a_k CH) > SU(a_{k+1}, \ldots, a_n/a_1, \ldots, a_k DH).$$

Proof. The proof is by induction on n. For $n = 1$ the result follows from Proposition 5.2 noticing that the arguments in the proposition work with the weaker assumption that the base sets C, D are H-independent sets.

Assume the result holds for n and consider $a_1, \ldots, a_{n+1} \in M$.

Assume that $tp(a_1, \ldots, a_n/D)$ forks over C or that $tp(a_{n+1}/a_1, \ldots, a_n D)$ forks over $a_1, \ldots, a_n C$. We apply the induction hypothesis and Proposition 5.2.

If $dim_{\mathrm{cl}}(a_1, \ldots, a_n/HD) < dim_{\mathrm{cl}}(a_1, \ldots, a_n/HC)$ or if $dim_{\mathrm{cl}}(a_{n+1}/a_1, \ldots, a_n HD) < dim_{\mathrm{cl}}(a_{n+1}/a_1, \ldots, a_n HC)$, then $dim_{\mathrm{cl}}(a_1, \ldots, a_{n+1}/HD) < dim_{\mathrm{cl}}(a_1, \ldots, a_{n+1}/HC)$ as we wanted.

Assume now that $dim_{\mathrm{cl}}(a_1, \ldots, a_n/HC) = dim_{\mathrm{cl}}(a_1, \ldots, a_n/HD)$ and that

$$dim_{\mathrm{cl}}(a_1, \ldots, a_{n+1}/HC) = dim_{\mathrm{cl}}(a_1, \ldots, a_{n+1}/HD).$$

If $HB(a_1, \ldots, a_n/C) \supsetneq HB(a_1, \ldots, a_n/D)$, then there is $h \in HB(a_1, \ldots, a_n/C)$ with $h \in D$. If $a_{n+1} \perp HDa_1, \ldots, a_n$, then $HB(a_1, \ldots, a_{n+1}/D) = HB(a_1, \ldots, a_n/D)$ and $HB(a_1, \ldots, a_{n+1}/C) = HB(a_1, \ldots, a_n/C)$.

So $HB(a_1, \ldots, a_{n+1}/C) \supsetneq HB(a_1, \ldots, a_{n+1}/D)$ as needed.

If $a_{n+1} \in \operatorname{cl}(HDa_1, \dots, a_n)$ then by the condition on dim_{cl} we must also have that $a_{n+1} \in \operatorname{cl}(HCa_1, \dots, a_n)$ and thus

$$HB(a_{n+1}/Da_1, \dots, a_n) \subset HB(a_{n+1}/Ca_1, \dots, a_n).$$

Thus we get again $HB(a_1, \dots, a_{n+1}/D) \subsetneq HB(a_1, \dots, a_{n+1}/D)$.

If $HB(a_1, \dots, a_n/C) = HB(a_1, \dots, a_n/D)$ and

$$HB(a_{n+1}/Da_1, \dots, a_n) \subsetneq HB(a_{n+1}/Ca_1, \dots, a_n)$$

then there is $h \in HB(a_{n+1}/Ca_1, \dots, a_n)$ with $h \in D$. Then $HB(a_1, \dots, a_{n+1}/C) \subsetneq HB(a_1, \dots, a_{n+1}/D)$.

Assume now that $dim_{\mathrm{cl}}(a_1, \dots, a_{n+1}/\operatorname{cl}(HD)) = dim_{\mathrm{cl}}(a_1, \dots, a_{n+1}/\operatorname{cl}(HD))$, $HB(a_1, \dots, a_{n+1}/C) = HB(a_1, \dots, a_{n+1}/D)$.

Case 1. $SU(a_{k+1}, \dots, a_n/a_1, \dots, a_kCH) > SU(a_{k+1}, \dots, a_n/a_1, \dots, a_kDH)$. Then if $a_{n+1} \in \operatorname{cl}(a_1, \dots, a_k, C, H)$ we also get by additivity of SU-rank that $SU(a_{k+1}, \dots, a_{n+1}/a_1, \dots, a_kCH) > SU(a_{k+1}, \dots, a_{n+1}/a_1, \dots, a_kDH)$ as desired.

If $a_{n+1} \notin \operatorname{cl}(a_1, \dots, a_k, C, H)$, then by the assumptions on dim_{cl} we also have $a_{n+1} \notin \operatorname{cl}(a_1, \dots, a_k, D, H)$ and $\operatorname{tp}(a_{n+1}/DHa_1, \dots, a_k)$ is orthogonal to

$$\operatorname{tp}(a_{k+1}, \dots, a_n/DHa_1, \dots, a_k)$$

and

$$SU(a_{k+1}, \dots, a_n/a_1, \dots, a_ka_{n+1}CH) > SU(a_{k+1}, \dots, a_n/a_1, \dots, a_ka_{n+1}DH),$$

as desired.

Case 2. $SU(a_{k+1}, \dots, a_n/a_1, \dots, a_kCH) = SU(a_{k+1}, \dots, a_n/a_1, \dots, a_kDH)$ and $SU(a_{n+1}/a_1 \dots, a_nCH) > SU(a_{n+1}/a_1 \dots, a_nDH)$. Then by additivity of SU-rank we have $SU(a_{k+1}, \dots, a_{n+1}/a_1, \dots, a_kCH) > SU(a_{k+1}, \dots, a_{n+1}/a_1, \dots, a_kDH)$ as desired.

The other direction is proved in a similar way. $\square$

We use the above result to give a different perspective on H-basis.

Lemma 5.4. *Let $(M, H(M))$ be an H-structure. Let $\vec{a} = (a_1, \dots, a_n) \in M$ and let $C \subset M$ be such that C is H-independent. Let $\vec{h}$ be a minimal tuple such that $\dim_{\mathrm{cl}}(\vec{a}/C\vec{h}) = \dim_{\mathrm{cl}}(\vec{a}/CH)$, then $\vec{h} = HB(\vec{a}/C)$.*

Proof. Write $\vec{a} = \vec{a}_1\vec{a}_2$, where $\vec{a}_1$ are independent generics over CH and $\vec{a}_2 \in \operatorname{cl}(\vec{a}_1CH)$. Choose $\vec{h}$ minimal so that $\vec{a}_2 \in \operatorname{cl}(\vec{a}_1C\vec{h})$. Then $\vec{a}_1$ are independent generics over CH and $SU(\vec{a}_2/\vec{a}_1C\vec{h}) < \omega$. Then $\operatorname{tp}(\vec{a}_1/C)$ is independent from H and $\operatorname{tp}(\vec{a}_2/\vec{a}_1C\vec{h})$ is orthogonal to H. We get $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{C\vec{h}} H$ and $HB(\vec{a}/C) \subset \vec{h}$. For the other direction, $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_{CHB(\vec{a}/C)} H$ implies that $\dim_{\mathrm{cl}}(\vec{a}/CHB(\vec{a}/C)) = \dim_{\mathrm{cl}}(\vec{a}/CH)$ and by minimality of $\vec{h}$ we get $HB(\vec{a}/C) \subset \vec{h}$. $\square$

We are interested in characterizing canonical bases. We start with the following result which holds also in the geometric setting:

Lemma 5.5. *Let (M, H) be a sufficiently saturated H-structure of T, $B \subset M$ an H-independent set, and $\vec{a} \in M$, $h = HB(\vec{a}/B)$ (viewed as an imaginary representing a finite set). Suppose $e \in \operatorname{acl}^{eq}(B)$ (in the original theory) is such that $\vec{a}h \mathop{\smile\!\!\!\!\!\!|}_e B$. Then $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_e^{ind} B$.*

Proof. We may assume that $\vec{a} = \vec{a}_1\vec{a}_2$, where $\vec{a}_1$ are generics over $B \cup H(M)$, $\vec{a}_2 \in \mathrm{cl}(H(M)B\vec{a}_1)$. Note that $\vec{a}_2 \in \mathrm{cl}(\vec{a}_1 Bh)$, so $\vec{a}h \mathop{\smile\!\!\!\!\!\!\;\mid}_e B$ implies that $\vec{a}_2 \in \mathrm{cl}(\vec{a}_1 eh)$, $SU(tp(\vec{a}_2/B\vec{a}_1 h)) = SU(tp(\vec{a}_2/e\vec{a}_1 h))$ and also $HB(\vec{a}/B) = HB(\vec{a}/e)$. Since $HB(\vec{a}/B) = HB(\vec{a}/e)$ and $\vec{a}h \mathop{\smile\!\!\!\!\!\!\;\mid}_e B$ by our characterization of forking in T^{ind} we get $\vec{a} \mathop{\smile\!\!\!\!\!\!\;\mid}_e^{ind} B$. □

Finally, the following result on canonical bases can be proved doing very small modifications to the argument presented in [5]:

Proposition 5.6. *Let (M, H) be a sufficiently saturated H-structure of T, $B \subset M$ an H-independent set, and $\vec{a} \in M$. Then $Cb_H(\vec{a}/B)$ and $Cb(\vec{a}HB(\vec{a}/B)/B))$ are interalgebraic.*

Proof. Let $e = Cb(\vec{a}HB(\vec{a}/B)/B))$. We saw in the previous lemma that $\vec{a} \mathop{\smile\!\!\!\!\!\!\;\mid}_e^{ind} B$ and thus $Cb_H(\vec{a}/B) \in \mathrm{acl}^{eq}(e)$.

We will now prove that e is in the algebraic closure of any Morley sequence in $\mathrm{stp}_H(\vec{a}/B)$.

Let $\{\vec{a}_i : i < \omega\}$ be an $\mathcal{L}_H$-Morley sequence in $\mathrm{tp}_H(\vec{a}/\mathrm{acl}_H^{eq}(B))$. Let $h_j = HB(\vec{a}_j/B)$ (viewed as an imaginary representing a finite set), so we have $h_j \in \mathrm{dcl}_H(\vec{a}_j B)$. Thus $\{\vec{a}_i h_i : i < \omega\}$ is also an $\mathcal{L}_H$-Morley sequence over B. This implies $h_j = HB(\vec{a}_j/B\vec{a}_{<j}h_{<j})$. We can write $\vec{a}_j = \vec{a}_{j1}\vec{a}_{j2}$ and hence by our characterization of forking in T^{eq} we have that $\vec{a}_{j1}h_j$ is an independent tuple of $\mathcal{L}$ generics over $B\vec{a}_{<j}h_{<j}$) and $SU(\vec{a}_{j2}/B\vec{a}_{<j}h_{<j}\vec{a}_{j1}h_j) = SU(\vec{a}_{j2}/B\vec{a}_{j1}h_j)$. Then it follows that $\mathrm{tp}(\vec{a}_j h_j/B\vec{a}_{<j}h_{<j})$ does not fork (in the sense of $\mathcal{L}$) over B. Thus, $\{\vec{a}_i h_i : i < \omega\}$ is also an $\mathcal{L}$-Morley sequence over B in $\mathrm{tp}(\vec{a}h/B)$. Since $\mathrm{tp}(\vec{a}_0 h_0/\{\vec{a}_i h_i : 0 < i < \omega\}B)$ is a free extension of $\mathrm{tp}(\vec{a}_0 h_0/\{\vec{a}_i h_i : 0 < i < \omega\})$ we also get that $e = Cb(\vec{a}_0 h_0/\{\vec{a}_i h_i : 0 < i < \omega\})$. It follows that $e \in \mathrm{acl}^{eq}(\{\vec{a}_i h_i : i < \omega\})$.

Since T^{ind} is supersimple there is $N \in \omega$ such that for all $n \geq N$, $\vec{a}_n \mathop{\smile\!\!\!\!\!\!\;\mid}_{\vec{a}_{<N}}^{ind} B$. By Proposition 4.7 $\mathrm{acl}_H(\vec{a}_{<N})$ is H-independent. By our characterization of non-forking, $HB(\vec{a}_n/B) = HB(\vec{a}_n/B\vec{a}_{<N}) = HB(\vec{a}_n/\mathrm{acl}_H(\vec{a}_{<N}))$ and in particular $h_n \in \mathrm{acl}_H(\vec{a}_i : i < \omega)$ for every $n \geq N$. We then get $e \in \mathrm{acl}_H^{eq}(\{\vec{a}_i : N \leq i < \omega\})$. Now, since $\{\vec{a}_i : i < \omega\}$ is a Morley sequence in $\mathrm{tp}_H(\vec{a}/\mathrm{acl}_H^{eq}(B))$, we have

$$\{\vec{a}_i : N \leq i < \omega\} \mathop{\smile\!\!\!\!\!\!\;\mid}_{Cb_H(\vec{a}/B)}^{ind} B,$$

and thus also

$$\{\vec{a}_i : N \leq i < \omega\} \mathop{\smile\!\!\!\!\!\!\;\mid}_{Cb_H(\vec{a}/B)}^{ind} e.$$

It follows that $e \in \mathrm{acl}_H^{eq}(Cb_H(\vec{a}/B))$, as needed. □

6. Ampleness

In this section we examine the relation between the ampleness of T and T^{ind}. In [4] it is shown an example of an one-based geometric theory T such that T^{ind} is not one-based. We follow the ideas on [7] to understand exactly when one-based is preserved and to show that non 2-ampleness is also preserved. In this section we will assume that T eliminates imaginaries.

Remark 6.1. *If T eliminates imaginaries then canonical bases are interalgebraic with real tuples. By Proposition 5.6 canonical bases in T^{ind} are also interalgebraic with real tuples. Hence T^{ind} has geometric elimination of imaginaries.*

Example 6.2. *Let G be an one-based stable group of U-rank ω and $T = Th(G)$. Notice that T^{ind} is again a stable theory so (M, H) is a stable group but clearly H is not a boolean combination of cosets of subgroups, so T^{ind} is not one-based.*

Definition 6.3. A pregeometry (X, cl) is *trivial* if for every $A \subset X$, $cl(A) = \bigcup_{a \in A} cl(a)$.

Notice that if G is a group of U-rank ω then cl is not trivial (take $a \mathop{\smile\!\!\!\!\!\mid} b$ both of rank ω and $c = a + b$, then $c \in \mathrm{cl}(a, b) \setminus \mathrm{cl}(a) \cup \mathrm{cl}(b)$).

Remark 6.4. *In the theory of the free pseudoplane (see example 3.2) the pregeometry generated by* cl *is trivial: for A algebraically closed and a a single element, $U(a/A) = d(a, A)$ where $d(a, A)$ is the minimum length of a path from a to an element of A (or ω if there is no path). If $b \in \mathrm{cl}(A)$ it means that there is a path to some element $a \in A$ so $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.*

We will now prove that one-basedness is only preserved in T^{ind} when the pregeometry cl is trivial. It is worth to notice that, unlike the U-rank 1 case, the triviality of cl does not imply that T is one-based. In fact, the theory of the free pseudoplane is the canonical example of a CM-trivial theory which is not one-based. This is the reason why the statement of the following proposition is a little bit different from [7].

Lemma 6.5. *If* cl *is trivial in T then for every $\vec{a}$ and for every $B = \mathrm{acl}_H(B)$,*

$$HB(\vec{a}/B) \subset HB(\vec{a}).$$

Proof. Let $h = HB(\vec{a}/B) = \{h_i | i \in I\}$. By minimality of H-bases for every $i \in I$ $\vec{a} \not\mathop{\smile\!\!\!\!\!\mid}_{Bh \setminus h_i} h_i$, then $h_i \in \mathrm{cl}(\vec{a}Bh \setminus h_i)$. As B is H-independent and $h_i \notin B$ then $h_i \mathop{\smile\!\!\!\!\!\mid} Bh \setminus h_i$, hence $h_i \notin \mathrm{cl}(Bh \setminus h_i)$. By triviality it means that $h_i \in \mathrm{cl}(a_i)$ for some $a_i \in \vec{a}$. By exchange property $a_i \in \mathrm{cl}(h_i)$, this implies $a_i \not\mathop{\smile\!\!\!\!\!\mid} h_i$ and $a_i \mathop{\smile\!\!\!\!\!\mid}_{h_i} H$ because $tp(a_i/h_i)$ is orthogonal to H. We conclude that $h_i = HB(a_i)$ and $HB(\vec{a}/B) = \{h_i | i \in I\} = \bigcup_{a_i \in A} HB(a_i) \subset HB(\vec{a})$. $\square$

Proposition 6.6. *Assume T is one-based, then T^{ind} is one-based if and only if* cl *is trivial in T.*

Proof of Proposition 6.6. ($\Leftarrow$) Assume cl is trivial, let $\vec{a}$ be a tuple, B an algebraic closed set in (M, H) and $\vec{h} = HB(\vec{a}/B)$. By the characterization of canonical bases, $\mathrm{acl}_H(cb_H(\vec{a}/B)) = \mathrm{acl}_H(cb(\vec{a}\vec{h}/B))$, as T is one-based, $cb(\vec{a}\vec{h}/B) \subset \mathrm{acl}(\vec{a}\vec{h})$. By the previous lemma, $\vec{h} \subset HB(\vec{a})$ then $cb_H(\vec{a}/B) \subset \mathrm{acl}_H(\vec{a}HB(\vec{a})) = \mathrm{acl}_H(\vec{a})$, i.e. T^{ind} is one-based.

($\Rightarrow$) Assume T^{ind} is one-based and cl is not trivial, then there are a tuple $\vec{a}$ and elements b and h such that $b \in \mathrm{cl}(\vec{a}h)$ and $b \notin \mathrm{cl}(\vec{a}) \cup \mathrm{cl}(h)$. We can take $\vec{a}$ cl independent tuple minimal with this property and, by the generalized extension property, we may assume that $\vec{a} \mathop{\smile\!\!\!\!\!\mid} H$. Moreover, as $h \notin \mathrm{cl}(\vec{a})$, we may assume also that h belongs to H by the generalized density property.

As $b \in \mathrm{cl}(\vec{a}h)$ and $\vec{a}h$ is H-independent, $tp(b/\vec{a}h)$ is orthogonal to H, i.e. $b \mathop{\smile\!\!\!\!\!\mid}_{h\vec{a}} H$. Recall that $b \not\mathop{\smile\!\!\!\!\!\mid}_{\vec{a}} h$ and h is a single element, then $h = HB(b/\vec{a})$. By hypothesis T^{ind} is one-based, then $\operatorname{acl}_H(cb_H(b/\vec{a})) = \operatorname{acl}_H(b) \cap \operatorname{acl}_H(\vec{a})$. Now, $\operatorname{acl}_H(\vec{a}) = \operatorname{acl}(\vec{a})$ as $\vec{a} \perp H$. On the other hand, as $\vec{a} \perp H$, and $b \perp_{h\vec{a}} H$ we have $b \perp_h H$. By hypothesis $b \notin \operatorname{cl}(h)$, hence $b \perp h$ (recall that b is a single element) and by transitivity $b \perp H$. So $HB(b) = \emptyset$ and $\operatorname{acl}_H(b) = \operatorname{acl}(b)$. This means $\operatorname{acl}_H(cb_H(b/\vec{a})) = \operatorname{acl}(b) \cap \operatorname{acl}(\vec{a})$.

Recall that $\operatorname{acl}_H(cb_H(b/\vec{a})) = \operatorname{acl}_H(cb(bh/\vec{a}))$. So a maximal cl-independent subset $\vec{d}$ of $cb(bh/\vec{a})$ satisfies that $b \in \operatorname{cl}(\vec{d}h)$ and $b \notin \operatorname{cl}(\vec{d}) \cup \operatorname{cl}(h)$. The minimality of the length of $\vec{a}$ yields $\operatorname{cl}(cb(bh/\vec{a})) = \operatorname{cl}(\vec{a})$, hence $\operatorname{cl}(\vec{a}) = \operatorname{cl}(\operatorname{acl}(a) \cap \operatorname{acl}(b)) \subset \operatorname{cl}(\vec{a}) \cap \operatorname{cl}(b)$, then $\vec{a} \in \operatorname{cl}(b)$ and $h \in \operatorname{cl}(\vec{a}b) \subset \operatorname{cl}(b)$. This is a contradiction. □

The notion of ampleness, defined by Pillay, captures forking complexity. He proved in [13] that a theory T is one-based if and only if is not 1-ample, a theory T is CM-trivial if and only if is not 2-ample. Moreover if T interprets a field then it is n-ample for every n.

Definition 6.7. A supersimple theory T is CM-trivial if for every tuple c and for every $A \subset B$, if $\operatorname{acl}^{eq}(cA) \cap \operatorname{acl}^{eq}(B) = \operatorname{acl}^{eq}(A)$ then $cb(c/A) \subset \operatorname{acl}^{eq}(cb(c/B))$

Definition 6.8. A supersimple theory T is *n-ample* if (possibly after naming some parameters) there exist tuples $a_0, ..., a_n$ in M^{eq} satisfying the following conditions:

For all $1 \leq i \leq n-1$.

(1) $a_{i+1} \perp_{a_i} a_{i-1} ... a_0$,

(2) $\operatorname{acl}^{eq}(a_0 ... a_{i-1} a_{i+1}) \cap \operatorname{acl}^{eq}(a_0 ... a_{i-1} a_i) = \operatorname{acl}^{eq}(a_0 ... a_{i-1})$.

(3) $a_n \not\perp_{\operatorname{acl}^{eq}(a_1) \cap \operatorname{acl}^{eq}(a_0)} a_0$.

Following [7] we prove that CM-triviality is preserved in T^{ind}. First we need the following lemma.

Lemma 6.9. *Let $A \subset B$, $A = \operatorname{acl}_H(A)$ y $B = \operatorname{acl}_H(B)$. If $\operatorname{acl}_H(cA) \cap B = A$ then $HB(c/A) \subset HB(c/B)$.*

Proof. It is clear that
$$HB(cA) \subset HB(cB).$$
By transitivity
$$HB(cA) = HB(c/A) \cup HB(A),$$
and the same with $HB(cB)$, hence
$$HB(c/A) \cup HB(A) \subset HB(c/B) \cup HB(B),$$
in particular $HB(c/A) \subset HB(c/B) \cup HB(B)$.

Now, if $HB(c/A) \cap H(B) = \emptyset$ we are done, but
$$HB(c/A) \cap HB(B) \subset \operatorname{acl}_H(cA) \cap B = A$$
and $HB(c/A) \cap A = \emptyset$. □

Proposition 6.10. *Let T be a SU-rank ω theory eliminating imaginaries, then T is CM-trivial if and only if T^{ind} is CM-trivial.*

Proof. Assume T is 2-ample. Let a_0, a_1, a_2 be tuples such that:

(1) $a_2 \underset{a_1}{\downarrow} a_0$,

(2) $\mathrm{acl}(a_0a_2) \cap \mathrm{acl}(a_0a_1) = \mathrm{acl}(a_0)$,

(3) $a_2 \underset{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)}{\not\downarrow} a_0$.

By the generalized extension property, there are a_0', a_1', a_2' such that $tp(a_0'a_1'a_2') = tp(a_0a_1a_2)$ and $a_0'a_1'a_2' \downarrow H$.

As the H-bases of any subset of $\{a_0'a_1'a_2'\}$ are empty, the algebraic closure in T^{ind} of any of these sets is the same as in T. So condition (2) holds in T^{ind}.

By the characterization of canonical bases, since H-bases are empty then condition (1) holds also in T^{ind}. But if

$$a_2' \underset{\mathrm{acl}_H(a_1')\cap\mathrm{acl}_H(a_0')}{\overset{H}{\downarrow}} a_0'$$

then

$$a_2' \underset{\mathrm{acl}(a_1')\cap\mathrm{acl}(a_0')}{\downarrow} a_0'.$$

This is a contradiction.

Assume T is not 2-ample, so it is CM-trivial. Let us see that T^{ind} is CM-trivial.

Let c be a tuple and $A \subset B$ be algebraically closed sets (in T^{ind}) such that $\mathrm{acl}_H(cA) \cap B = A$. Define $h = HB(c/A)$, $h' = HB(c/B)$ and $c' = ch$. By Proposition 5.6 we have $\mathrm{acl}_H(cb_H(c/A)) = \mathrm{acl}_H(cb(ch/A))$ and by Lemma 6.9 $h \subset h'$. Note that $\mathrm{acl}(c'A) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$ because $\mathrm{acl}(c'A) \subset \mathrm{acl}_H(cA)$, $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$. So, by CM-triviality of T, $cb(c'/A) \subset \mathrm{acl}(cb(c'/B))$. Recall that $c' = ch$. Hence

$$\begin{aligned}
\mathrm{acl}_H(cb_H(c/A)) &= \mathrm{acl}_H(cb(ch/A)) \\
&\subset \mathrm{acl}_H(cb(ch/B)) \\
&\subset \mathrm{acl}_H(cb(ch'/B)) \\
&= \mathrm{acl}_H(cb_H(c/B)).
\end{aligned}$$

□

We can adapt the previous proof in order to prove that if T^{ind} is n-ample then T is n-ample for every n. In [7] the converse has been proved for SU-rank 1 theories with a predicate, but we could not adapt that proof to this context.

7. Geometry modulo H in the one-based case

In this section we consider the case when T is one-based, and follow the proofs of Theorem 5.13 [15] and the results of Section 6 of [15], and Section 4 of [4], to study the geometry induced by cl localized at $H(M)$. Many of the proofs are nearly identical to the ones from [15] and [4], we include them for completeness.

Let (M, H) be a sufficiently saturated model of T^{ind}. Let cl_H be the localization of the operator cl at $H(M)$, i.e. $\mathrm{cl}_H(A) = \mathrm{cl}(A \cup H(M))$. Thus, $a \in \mathrm{cl}_H(B)$ means $SU(a/B \cup H(M)) < \omega$.

Proposition 7.1. *Suppose T is one-based. Then the pregeometry (M, cl_H) is modular.*

Proof. It suffices to show that for any $a, b \in M$ and a small set $C \subset M$, if $a \in \mathrm{cl}_H(bC)$ then there exists $d \in \mathrm{cl}_H(C)$ such that $a \in \mathrm{cl}_H(bd)$. We may assume that $a, b \notin \mathrm{cl}_H(C)$. Let $\vec{h} \in H(M)$ be finite such that $a \in \mathrm{cl}(bC\vec{h})$. Let $e = Cb(ab/C\vec{h})$. Thus, by one-basedness of T, $e \in \mathrm{acl}^{eq}(ab) \cap \mathrm{acl}^{eq}(C\vec{h})$. By the density property, there is $b' \models \mathrm{tp}(b/\mathrm{acl}^{eq}(C\vec{h}))$, $b' \in H(M)$. Take $a' \in M$ such that $\mathrm{tp}(a'b'/\mathrm{acl}^{eq}(C\vec{h})) = \mathrm{tp}(ab/\mathrm{acl}^{eq}(C\vec{h}))$. Then $e \in \mathrm{acl}^{eq}(a'b')$. Clearly, $a' \in \mathrm{cl}(b'C\vec{h}) \subset \mathrm{cl}_H(C)$. Also, $ab \mathop{\smile\!\!\!\!\!\!|}_e C\vec{h}$ implies $SU(a/be) = SU(a/bC\vec{h}) < \omega$. Since $e \in \mathrm{acl}^{eq}(a'b')$, we have $SU(a/ba'b') \leq SU(a/be) < \omega$. Since $b' \in H(M)$, this implies $a \in \mathrm{cl}_H(ba')$. Hence, taking $d = a'$, we have $d \in \mathrm{cl}_H(C)$ and $a \in \mathrm{cl}_H(bd)$, as needed. ☐

Let (M^*, cl^*) be the geometry associated with (M, cl_H) (i.e. M^* is the set $M \backslash \mathrm{cl}_H(\emptyset)$ modulo the relation $\mathrm{cl}_H(x) = \mathrm{cl}_H(y)$) . For any $a \notin \mathrm{cl}_H(\emptyset)$, let a^* be the class of a modulo the relation $\mathrm{cl}_H(x) = \mathrm{cl}_H(y)$. Define the relation $\sim$ by

$$a^* \sim b^* \iff |\mathrm{cl}^*(a^*, b^*)| \geq 3 \text{ or } a^* = b^*.$$

Lemma 7.2. *For any $a, b \in M$, $a^* \sim b^*$ if and only if there exist $d_1, \ldots, d_n \in M$ such that*

$$a^* \in \mathrm{cl}^*(b^* d_1^* \ldots d_n^*) \backslash \mathrm{cl}^*(d_1^* \ldots d_n^*).$$

Proof. The "only if" direction is clear. For the "if" direction, suppose $a^* \neq b^*$ and $a^* \in \mathrm{cl}^*(b^* d_1^* \ldots d_n^*) \backslash \mathrm{cl}^*(d_1^* \ldots d_n^*)$. We may assume that $n \geq 1$ is minimal such. Then $a \in \mathrm{cl}^*(bd_1 \ldots d_n h_1 \ldots h_k)$ for some $h_1, \ldots, h_k \in H(M)$. We may assume that k is minimal such. Then the tuple $abd_2 \ldots d_n h_1 \ldots h_k$ is cl-independent. By the density property, we can find $d_2', \ldots, d_n' \in H(M)$ such that $\mathrm{tp}(d_2', \ldots, d_n'/ab\vec{h}) = \mathrm{tp}(d_2, \ldots, d_n/ab\vec{h})$. Let $d_1' \in M$ be such that

$$\mathrm{tp}(d_1', d_2', \ldots, d_n'/ab\vec{h}) = \mathrm{tp}(d_1, d_2, \ldots, d_n/ab\vec{h}).$$

Then $d_1' \notin \mathrm{cl}_H(\emptyset)$ and $(d_1')^* \in \mathrm{cl}^*(a^*, b^*)$, while $(d_1')^* \neq a^*, b^*$. Thus, $|\mathrm{cl}^*(a^*, b^*)| \geq 3$, as needed.

☐

Lemma 7.3. *The relation $\sim$ is an equivalence on M^*.*

Proof. Reflexivity and symmetry are clear. For transitivity, assume $a^* \sim b^* \sim c^*$, with all three distinct. Then there exist $d_1^* \in \mathrm{cl}^*(a^* b^*) \backslash \{a^*, b^*\}$ and $d_2^* \in \mathrm{cl}^*(b^*, c^*) \backslash \{b^*, c^*\}$. If $d_1^* = d_2^*$, then $c^* \in \mathrm{cl}^*(b^*, d_2^*) = \mathrm{cl}^*(b^*, d_1^*) = \mathrm{cl}^*(a^*, d_1^*)$, and therefore $d_1^* = d_2^* \in \mathrm{cl}^*(a^*, c^*) \backslash \{a^*, c^*\}$, hence $a^* \sim c^*$.

Now, assume that $d_1^* \neq d_2^*$ and $a^* \in \mathrm{cl}^*(d_1^*, d_2^*)$. If $a^* = d_2^*$, then b^* witnesses $a^* \sim c^*$. If $a^* \neq d_2^*$, then $d_2^* \in \mathrm{cl}^*(a^*, d_1^*)$. We also have $b^* \in \mathrm{cl}^*(a^*, d_1^*)$, $c^* \in \mathrm{cl}^*(b^*, d_2^*)$. Thus, $c^* \in \mathrm{cl}^*(a^*, d_1^*)$. If $c^* = d_1^*$, b^* witnesses $a^* \sim c^*$. If $c^* \neq d_1^*$, then d_1^* witnesses $a^* \sim c^*$. Finally, assume that $d_1^* \neq d_2^*$ and neither $a^* \notin \mathrm{cl}^*(d_1^*, d_2^*)$. Then

$$a^* \in \mathrm{cl}^*(c^* d_1 d_2^*) \backslash \mathrm{cl}^*(d_1^* d_2^*).$$

Thus, by Lemma 7.2, $a^* \sim c^*$.

☐

For any $a^* \in M^*$ let $[a^*]$ denote the $\sim$-class of a^*.

Lemma 7.4. *The $\sim$-classes are closed in the sense of* cl^**, i.e. for any* $a^* \in M^*$*, we have* $\mathrm{cl}^*([a^*]) = [a^*]$.

Proof. Assume $c^* \in \mathrm{cl}^*(b_1^*, \dots, b_n^*)$, $\vec{b}^* = (b_1^*, \dots, b_n^*) \in [a^*]$ minimal such tuple, and $n > 1$ (if $n = 1$, we have $c^* = b_1^*$). Then $b_1^* \dots b_{n-1}^*$ witnesses $c^* \sim b_n^*$, by Lemma 7.2. $\square$

For any geometry (X, Cl), a non-empty subset of X, with the closure operator induced by Cl, is referred to as a *subgeometry* of (X, Cl). Clearly, a subgeometry is itself a geometry. Next lemma shows that $\sim$ splits (M^*, cl^*) into disjoint subgeometries of the form $([a^*], \mathrm{cl}^*)$, with no "interaction" between them.

Lemma 7.5. *For any* $A \subset M^*$*,* $\mathrm{cl}^*(A) = \bigcup_{[a^*] \in M^*/\sim} \mathrm{cl}^*(A \cap [a^*])$.

Proof. Suppose $c^* \in \mathrm{cl}^*(A)$, and $a_1^*, \dots, a_n^* \in A$ is a tuple such that $c \in \mathrm{cl}^*(a_1^*, \dots, a_n^*)$, and n is minimal such. It suffices to show that a_i^* all come from the same $\sim$-class. If $n = 1$, we are done. Suppose $n > 1$. Then $c^* a_3^* \dots a_n^*$ witnesses $a_1^* \sim a_2^*$ by Lemma 7.2. Similarly, $a_1^* \sim a_i^*$ for all $2 < i \leq n$. Thus, all a_i^* belong to the same $\sim$-class. $\square$

Next, we will show that the $\sim$-classes are either singletons or infinite dimensional (as geometries).

Lemma 7.6. *If* $|[a^*]| > 1$*, then* $\dim([a^*])$ *is infinite.*

Proof. Suppose there exists $b^* \sim a^*$, $b^* \neq a^*$. Let $c^* \in \mathrm{cl}^*(a^*, b^*) \backslash \{a^*, b^*\}$. Let $a, b, c \in M$ be representatives of the classes a^*, b^* and c^* modulo the relation $\mathrm{cl}_H(x) = \mathrm{cl}_H(y)$, respectively.

Then $SU(a/H(M)) = SU(b/aH(M)) = \omega$. By the extension property, we can find $b_i \models \mathrm{tp}(b/a)$, $i \in \omega$, independent over $aH(M)$. Then, by Lemma 2.7, $\mathrm{tp}_H(b_i/a) = \mathrm{tp}_H(b/a)$ for all $i \in \omega$. Also, b_i are cl_H-independent over a. Let c_i be such that $\mathrm{tp}_H(b_i c_i/a) = \mathrm{tp}_H(bc/a)$ for $i \in \omega$. Passing to the geometry (M^*, cl^*), we get $b_i^* \sim a^*$ witnessed by c_i^*, $i \in \omega$, with b_i cl^*-independent over a^*. This shows that that $([a^*], \mathrm{cl}^*)$ is infinite dimensional. $\square$

Recall the following classical fact (see [11]) about projective geometries.

Fact 7.7. *A non-trivial modular geometry of dimension* ≥ 4 *in which any closed set of dimension* 2 *has size* ≥ 3 *is a projective geometry over some division ring.*

Lemma 7.8. *If* T *is one-based and* $|[a^*]| > 1$ *, the geometry* $([a^*], cl)$ *is an infinite dimensional projective geometry over some division ring.*

Proof. By Proposition 7.1, (M^*, cl^*) is modular. By Lemma 7.5, $[a^*]$ is closed in (M^*, cl^*), and hence $([a^*], \mathrm{cl}^*)$ is also modular. Since $|[a^*]| > 1$, $([a^*], \mathrm{cl}^*)$ is non-trivial (there are two distinct point having a third one in its closure). Thus, the statement follows by Fact 7.7 and the definition of $\sim$. $\square$

We are now ready to prove the characterization of the geometry of cl_H, as well as the original geometry of cl in the case when T is one-based.

Proposition 7.9. *Suppose* T *is a one-based supersimple theory of SU-rank* ω*,* (N, H) *a sufficiently (e.g.* $|T|^+$*-) saturated models of* T^{ind}*, and* M *a small model of* T *(e.g. of size* $|T|$*). Then*

(1) The geometry (N^*, cl^*) of cl_H in (N, H) is a disjoint union of infinite dimensional projective geometries over division rings and/or a trivial geometry.
(2) The geometry of the closure operator cl *in* M *is a disjoint union of subgeometries of projective geometries over division rings.*

Proof. (1) Follows by Lemmas 7.5, 7.6 and 7.8.
(2) By Lemma 2.6, any structure of the form (M, H) where $M \models T$, and $H(M)$ is an independent set of generics, can be embedded, in an H-independent way, in a sufficiently saturated H-structure. Thus we may assume that $(M, \emptyset) \subset (N, H)$ with $M \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} H(N)$. Then cl-independence over $\emptyset$ in M coincides with cl-independence in N over $H(N)$, i.e. cl_H-independence. Thus, we have a natural embedding of the associated geometry of (M, cl) into (N^*, cl^*). The result now follows by (1). $\square$

Remark 7.10. *The previous proposition also holds with the weaker assumption that the pregeometry (N, cl_H) is modular instead of asking that is one-based. All the proofs depend on the properties of the closure operator, not the properties of forking in the full structure.*

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